On the diagonals of a Rees algebra

Olga Lavila–Vidal

PhD Thesis

Universitat de Barcelona (1999)

February 1, 2008

Contents

Introduction

The aim of this work is to study the ring-theoretic properties of the diagonals of a Rees algebra, which from a geometric point of view are the homogeneous coordinate rings of embeddings of blow-ups of projective varieties along a subvariety. First we are going to introduce the subject and the main problems. After that we shall review the known results about these problems, and finally we will give a summary of the contents and results obtained in this work.

Let A be a noetherian graded algebra generated over a field k by homogeneous elements of degree 1, that is, A has a presentation $A = k[X_1, \ldots, X_n]/K = k[x_1, \ldots, x_n]$, where K is a homogeneous ideal of the polynomial ring $k[X_1, \ldots, X_n]$ with the usual grading. Given a homogeneous ideal I of A, let X be the projective variety obtained by blowing-up the projective scheme $Y = \text{Proj}\ (A)$ along the sheaf of ideals $\mathcal{I} = \widetilde{I}$, that is, $X = \mathcal{P}roj(\bigoplus_{n\geq 0} \mathcal{I}^n)$. For a given $c \in \mathbb{Z}$, let us denote by I_c the c-graded component of I. If I is generated by forms of degree less or equal than d, then $(I^e)_c$ corresponds to a complete linear system on X very ample for $c \geq de+1$ which embeds X in a projective space $X \cong \text{Proj}\ (k[(I^e)_c]) \subset \mathbb{P}_k^{N-1}$, with $N = \dim_k (I^e)_c$ [CH, Lemma 1.1].

Our main purpose is to study the arithmetic properties of the k-algebras $k[(I^e)_c]$, where c, e are positive integers and I is any homogeneous ideal of A. This problem was first started in the work by A. Gimigliano [Gi], A. Geramita and A. Gimigliano [GG], and A. Geramita, A. Gimigliano and B. Harbourne [GGH] who treated similar problems for the rational projective surfaces which arise as embeddings of blow-ups of a projective plane at a set of distinct points.

Let k be an algebraically closed field and $s = \binom{d+1}{2}$, $d \geq 2$. In [Gi] the particular case of the blow-up of $\mathbb{P}_k^2$ at a set of s different points $P_1, \ldots, P_s$ which do not lie on a curve of degree $d-1$ and such that there is no subset of

d points on a line (if $d \geq 3$) is studied in detail. In this case, the defining ideal I of the set of points is generated by forms of degree d and the rational maps defined by the linear systems I_c give embeddings of the blow-up for $c \geq d$. In the case $c = d$ the surface obtained is called *White Surface*, and for $c = d+1$ *Room Surface*. It is then shown that White surfaces are contained in $\mathbb{P}^d_k$ as surfaces of degree $\binom{d}{2}$ with defining ideal generated by the maximal minors of a $3 \times d$ matrix of linear forms. In particular, $k[I_d]$ is Cohen-Macaulay and it has a resolution given by the Eagon-Northcott complex [Gi, Proposition 1.1]. On the other hand, Room Surfaces are arithmetically Cohen-Macaulay [GG, Theorem B] with defining ideal generated by quadrics [GG, Theorem 1.2].

This detailed study of White and Room Surfaces is the first step to consider the following more general case. Let $P_1, \ldots, P_s$ be s distinct points in $\mathbb{P}^2_k$, with k an algebraically closed field, let I be its defining ideal and $d = \operatorname{reg}(I)$ the regularity of I. Assume that the points do not lie on a curve of degree $d-1$ and that there is no subset of d points on a line. Then the linear systems I_c give embeddings of the blowing-up of $\mathbb{P}^2_k$ at this set of points for $c \geq d$. The resultant surfaces are arithmetically Cohen-Macaulay [GG, Theorem B] and its defining ideal is defined by quadrics if $c \geq d+1$ [GG, Theorem 2.1].

Even more generally, A. Geramita, A. Gimigliano and Y. Pitteloud [GGP] consider the blow-up of $\mathbb{P}^n_k$ along an ideal of fat points, with k an algebraically closed field of characteristic zero. Given a set of points $P_1, \ldots, P_s \in \mathbb{P}^n_k$, let $\mathcal{P}_1, \ldots, \mathcal{P}_s \subset k[X_0, \ldots, X_n]$ be their defining ideals, and let us take ideals of the type $I = \mathcal{P}_1^{m_1} \cap \ldots \cap \mathcal{P}_s^{m_s}$, with $m_1, \ldots, m_s \in \mathbb{Z}_{\geq 1}$. Then one may study the projective varieties obtained by embeddings of the blow-up of $\mathbb{P}^n_k$ along $\mathcal{I}$ via the linear systems corresponding to the graded pieces of I, whenever these linear systems are very ample. Let $d = \operatorname{reg}(I)$, and let us assume that there are not d points on a line. Then the linear systems I_c are very ample for $c \geq d$, and the varieties obtained via these embeddings are projectively normal [GGP, Proposition 2.2] and arithmetically Cohen-Macaulay [GGP, Theorem 2.4].

A new point of view to treat these questions was introduced by A. Simis, N.V. Trung and G. Valla in [STV], and later followed by A. Conca, J. Herzog, N.V. Trung and G. Valla in [CHTV], to study the more general problem of the blow-up of a projective space along an arbitrary subvariety. If I is a homogeneous ideal of A, let us consider the Rees algebra $R_A(I) = \bigoplus_{n \geq 0} I^n \cong$

$A[It] \subset A[t]$ of I with the natural bigrading given by

$$R_A(I)_{(i,j)} = (I^j)_i.$$

The crucial point now is that all the algebras $k[(I^e)_c]$ are subalgebras of the Rees algebra in a natural way. To describe this relationship we need to introduce the diagonal functor.

Given positive integers c, e, the (c, e)-*diagonal* of $\mathbb{Z}^2$ is the set

$$\Delta := \{(cs, es) \mid s \in \mathbb{Z}\}.$$

For any bigraded algebra $S = \bigoplus_{(i,j)\in\mathbb{Z}^2} S_{(i,j)}$, the *diagonal subalgebra* of S along Δ is the graded algebra

$$S_\Delta := \bigoplus_{s\in\mathbb{Z}} S_{(cs,es)}.$$

Similarly we may define the diagonal of a bigraded S-module L along Δ as the graded S_Δ-module

$$L_\Delta := \bigoplus_{s\in\mathbb{Z}} L_{(cs,es)}.$$

So we have an exact functor

$$(\ \ )_\Delta : M^2(S) \to M^1(S_\Delta),$$

where $M^2(S)$, $M^1(S_\Delta)$ denote the categories of bigraded S-modules and graded S_Δ-modules respectively.

Now we may give a description of the rings $k[(I^e)_c]$ as diagonals of the Rees algebra in the following way: By taking Δ to be the (c, e)-diagonal of $\mathbb{Z}^2$ with $c \geq de + 1$, we have

$$R_A(I)_\Delta = \bigoplus_{s\geq 0} (I^{es})_{cs} = k[(I^e)_c].$$

This observation allows an algebraic approach to study the rings $k[(I^e)_c]$ via the diagonals of $R_A(I)$. This is the starting point in [STV] to study the case of homogeneous ideals of the polynomial ring generated by forms of the same degree, and later in [CHTV] to study arbitrary homogeneous ideals of the polynomial ring. By paraphrasing [STV]: *One is to believe that the algebraic approach via the diagonals of the Rees algebra may throw further light not only on the study of embedded rational surfaces obtained by blowing-up a*

set of points in $\mathbb{P}^2_k$ but also of the embedded rational n-folds obtained, more generally, by blowing-up $\mathbb{P}^n_k$ along some special smooth subvariety. On the other hand, the diagonals of any standard bigraded algebra defined over a local ring have also been studied by E. Hyry [Hy] by using both an algebraic approach and a geometric approach. Finally, S.D. Cutkosky and J. Herzog [CH] have studied the diagonals of the Rees algebra of a homogeneous ideal in a general graded k-algebra.

Next we are going to expose the main results of those works.

The main contribution of A. Simis et al. [STV] to the problems considered by A. Geramita et al. is the algebraic approach via the diagonal of a bigraded algebra, a notion which generalizes the Segre product of graded algebras. Given algebraic varieties $V \subset \mathbb{P}^{n-1}_k$, $W \subset \mathbb{P}^{r-1}_k$ with homogeneous coordinate rings R_1, R_2, the image of $V \times W \subset \mathbb{P}^{n-1}_k \times \mathbb{P}^{r-1}_k$ under the Segre embedding

$$\mathbb{P}^{n-1}_k \times \mathbb{P}^{r-1}_k \hookrightarrow \mathbb{P}^{nr-1}_k$$

is a variety with homogeneous coordinate ring the Segre product of R_1 and R_2:

$$R_1 \underline{\otimes}_k R_2 = \bigoplus_{u \in \mathbb{N}} (R_1)_u \otimes_k (R_2)_u.$$

Given a standard bigraded k-algebra $R = \bigoplus_{(u,v)\in\mathbb{N}^2} R_{(u,v)}$, its diagonal R_Δ is defined as $R_\Delta = \bigoplus_{u\in\mathbb{N}} R_{(u,u)}$ (that is, the $(1,1)$-diagonal). By considering the tensor product $R = R_1 \otimes_k R_2$ bigraded by means of $R_{(u,v)} = (R_1)_u \otimes_k (R_2)_v$, we have that $R_\Delta = R_1 \underline{\otimes}_k R_2$. Classically R is taken to be the bihomogeneous coordinate ring of a projective subvariety of $\mathbb{P}^{n-1}_k \times \mathbb{P}^{r-1}_k$, and R_Δ is then the homogeneous coordinate ring of its image via the Segre embedding.

In the first section of [STV], a relation between the presentations, the dimensions and the multiplicities of a standard bigraded k-algebra R and its diagonal R_Δ is obtained. The key for proving these results is the existence of the Hilbert polynomial of a standard bigraded k-algebra and the characterization of its degree, due to D. Katz et al. [KMV] and M. Herrmann et al. [HHRT] among others. Similarly to the graded case, one may define in this case the irrelevant ideal, the irrelevant primes and the biprojective scheme associated to a standard bigraded k-algebra.

After that, it is studied the behaviour of the normality and the Cohen-Macaulay property by taking diagonals. Since there is a Reynolds operator

from R to R_Δ, one immediately gets that the normality of R will be inherited by its diagonal R_Δ. With respect to the Cohen-Macaulayness, the strategy is to reduce the problem to a special situation where the diagonal subalgebra becomes a Segre product, case in which it is known a criterion for the Cohen-Macaulayness.

These results are then applied to the study of the Rees algebra $R_A(I)$ of a homogeneous ideal $I \subset A = k[X_1, \ldots, X_n]$ generated by forms of the same degree d (*equigenerated* ideals). In this situation, the Rees algebra can be bigraded so that becomes standard by means of

$$R_A(I)_{(i,j)} = (I^j)_{i+dj},$$

and then $R_A(I)_\Delta = k[I_{d+1}]$. Mainly, two classes of ideals are then considered in detail: For complete intersection ideals generated by a regular sequence of r forms of degree d it is shown that $k[I_{d+1}]$ is a Cohen-Macaulay ring if $(r-1)d < n$, while $k[I_{d+1}]$ is not a Cohen-Macaulay ring if $(r-1)d > n$ [STV, Theorem 3.7]; for straightening closed ideals under some restrictions it is shown that $k[I_{d+1}]$ is a Cohen-Macaulay ring [STV, Theorem 3.13]. This second class of ideals includes for instance the determinantal ideals generated by the maximal minors of a generic matrix.

As a natural sequel of the work above, A. Conca et al. study in [CHTV] the diagonals R_Δ of a bigraded k-algebra R for $\Delta = (c, e)$, with c, e positive integers. The main problem considered there is to find suitable conditions on R such that certain algebraic properties of R are inherited by some diagonal R_Δ, mostly with respect to the Cohen-Macaulay property and the Koszul property. Their goal is to apply the results to the case of a standard bigraded k-algebra or the Rees algebra of any homogeneous ideal I of $A = k[X_1, \ldots, X_n]$. In the first case, R has a presentation as a quotient of a polynomial ring $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ endowed with the grading given by $\deg(X_i) = (1, 0)$, $\deg(Y_j) = (0, 1)$. As for the Rees algebra, if I is generated by forms $f_1, \ldots, f_r$ of degrees $d_1, \ldots, d_r$ respectively, we have a natural bigraded epimorphism

$$\begin{array}{ccc} S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_r] & \longrightarrow & R = R_A(I) \\ X_i & \mapsto & X_i \\ Y_j & \mapsto & f_j t \end{array}$$

where $\deg(X_i) = (1, 0)$, $\deg(Y_j) = (d_j, 1)$. Therefore, by working in the category of bigraded S-modules for $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ the polyno-

mial ring with $\deg(X_i) = (1,0)$, $\deg(Y_j) = (d_j, 1)$, $d_1, \ldots, d_r \geq 0$, one may study both cases at the same time. Let us denote by $\mathcal{M}$ and $m = \mathcal{M}_\Delta$ the homogeneous maximal ideals of S and S_Δ respectively. Denoting by $d = \max\{d_1, \ldots, d_r\}$, we will consider diagonals $\Delta = (c, e)$ with $c \geq de + 1$.

Since the arithmetic properties of a module can be often characterized in terms of its local cohomology modules, it is of interest to study the local cohomology of the diagonals L_Δ of any finitely generated bigraded S-module L. This is done from the bigraded minimal free resolution of L over S: Let

$$0 \to D_l \to \ldots \to D_0 \to L \to 0$$

with $D_p = \bigoplus_{(a,b)\in\Omega_p} S(a,b)$ be the bigraded minimal free resolution of L over S. By taking diagonals one gets a graded resolution of L_Δ

$$0 \to (D_l)_\Delta \to \ldots \to (D_0)_\Delta \to L_\Delta \to 0,$$

with $(D_p)_\Delta = \bigoplus_{(a,b)\in\Omega_p} S(a,b)_\Delta$. The first step is then the computation of the local cohomology modules of the S_Δ-modules $S(a,b)_\Delta$, which is done in the frame of a more general study about the local cohomology of the Segre product of two bigraded k-algebras. In particular, it is obtained a criterion for the Cohen-Macaulay property of $S(a,b)_\Delta$ by means of a, b and Δ. We say that the resolution of L is good if every module $(D_p)_\Delta$ is Cohen-Macaulay for large diagonals Δ. Then it is stated the following theorem:

Theorem [CHTV, Theorem 3.6, Lemma 3.8] *Assume $n \geq r$. For any finitely generated bigraded S-module L, there exists a canonical morphism*

$$\varphi_L^q : H_m^q(L_\Delta) \to H_{\mathcal{M}}^{q+1}(L)_\Delta, \ \forall q \geq 0$$

such that

(i) φ_L^q is an isomorphism for $q > n$.

(ii) φ_L^q is a quasi-isomorphism for $q \geq 0$.

(iii) If L has a good resolution, φ_L^q is an isomorphism for large diagonals.

As a corollary one gets necessary and sufficient conditions for the existence of Cohen-Macaulay or Buchsbaum diagonals L_Δ of L in terms of the graded pieces of the local cohomology modules of L.

Given a standard bigraded k-algebra R, one may define the graded k-subalgebras $\mathcal{R}_1 = \bigoplus_{i\in\mathbb{N}} R_{(i,0)}$, $\mathcal{R}_2 = \bigoplus_{j\in\mathbb{N}} R_{(0,j)}$. The following result gives a criterion for the Cohen-Macaulay property of the diagonals of R by means of $\mathcal{R}_1$ and $\mathcal{R}_2$. Namely,

Theorem [CHTV, Theorem 3.11] *Let R be a standard bigraded Cohen-Macaulay k-algebra. If the shifts in the resolutions of $\mathcal{R}_1$ and $\mathcal{R}_2$ are greater than $-n$ and $-r$ respectively, then R_Δ is Cohen-Macaulay for large Δ .*

In particular, they get the following corollary:

Corollary [CHTV, Corollary 3.12] *Let R be a standard bigraded Cohen-Macaulay k-algebra. If $\mathcal{R}_1$, $\mathcal{R}_2$ are Cohen-Macaulay with $a(\mathcal{R}_1), a(\mathcal{R}_2) < 0$, then R_Δ is Cohen-Macaulay for large Δ .*

This result applied to Rees algebras of equigenerated ideals gives a criterion for the Cohen-Macaulay property of their diagonals.

Furthermore, the study done in [STV] for the $(1,1)$-diagonal of the Rees algebra of an equigenerated complete intersection ideal is completed and extended to any complete intersection ideal and any diagonal, by determining exactly which are the Cohen-Macaulay diagonals. This is the only case where non equigenerated ideals are considered.

Theorem [CHTV, Theorem 4.6] *Let $I \subset A = k[X_1, \ldots, X_n]$ be a homogeneous complete intersection ideal minimally generated by r forms of degrees $d_1, \ldots, d_r$. Set $u = \sum_{j=1}^{r} d_j$. For $c \geq de + 1$, $k[(I^e)_c]$ is a Cohen-Macaulay ring if and only if $c > d(e-1) + u - n$.*

About the Cohen-Macaulay property of the diagonals of a Rees algebra is conjectured the following fact:

Conjecture *Let $I \subset A = k[X_1, \ldots, X_n]$ be a homogeneous ideal. If $R_A(I)$ is a Cohen-Macaulay ring, then there exists a diagonal Δ such that $R_A(I)_\Delta$ is a Cohen-Macaulay ring.*

With respect to the Gorenstein property, there is just one statement referred to the diagonals of the Rees algebra of a homogeneous ideal generated by a regular sequence of length 2.

Proposition [CHTV, Corollary 4.7] *Let $I \subset A = k[X_1, \ldots, X_n]$ be a homogeneous complete intersection ideal minimally generated by two forms of degree $d_1 \leq d_2$. If $n \geq d_2 + 1$, $k[I_n]$ is a Gorenstein ring with a-invariant -1.*

Finally, it is shown that large diagonals of the Rees algebra are always Koszul:

Theorem [CHTV, Corollary 6.9] *Let $I \subset A = k[X_1, \ldots, X_n]$ be a homogeneous ideal generated by forms of degree $\leq d$. Then there exist integers a, b such that $k[(I^e)_{c+de}]$ is Koszul for all $c \geq a$ and $e \geq b$.*

Under a slightly different setting, E. Hyry [Hy] is concerned with comparing the Cohen-Macaulay property of the biRees algebra $R_A(I, J)$ with the Cohen-Macaulay property of the Rees algebra $R_A(IJ)$, where $I, J \subset A$ are ideals of positive height in a local ring. To this end, he studies the $\Delta = (1,1)$-diagonal of any standard bigraded ring R defined over a local ring. The main result [Hy, Theorem 2.5] gives necessary and sufficient conditions for the Cohen-Macaulayness of a standard bigraded ring R with negative a-invariants by means of the local cohomology of the modules $R(p,0)_\Delta$ and $R(0,p)_\Delta$ ($p \in \mathbb{N}$). In particular, it provides sufficient conditions on R so that the Cohen-Macaulay property is carried from R to R_Δ:

Theorem *Let R be a standard bigraded ring defined over a local ring. Suppose that $\dim \mathcal{R}_1, \dim \mathcal{R}_2 < \dim R$ and $a^1(R), a^2(R) < 0$. If R is Cohen-Macaulay, then so is R_Δ for $\Delta = (1,1)$.*

Now let A be a noetherian graded k-algebra generated in degree 1 and let $I \subset A$ be a homogeneous ideal. The general problem of studying the embeddings of the blow-up $X = \mathcal{P}roj(\bigoplus_{n \geq 0} \mathcal{I}^n)$ of the projective scheme $Y = \text{Proj}\ (A)$ along the sheaf of ideals $\mathcal{I} = \widetilde{I}$ given by the graded pieces of I is treated by S.D. Cutkosky and J. Herzog [CH]. They are mainly concerned with the existence of an integer f such that $k[(I^e)_c]$ is Cohen-Macaulay for all $e > 0$ and $c \geq ef$. The first example considered is the blow-up of a smooth projective variety Y along a regular ideal in a field of characteristic zero, where the Kodaira Vanishing Theorem can be used to prove:

Theorem [CH, Theorem 1.6] *Suppose that k has characteristic zero, A is Cohen-Macaulay, Y is smooth, I is equidimensional and* Proj (A/I) *is smooth. Then there exists a positive integer f such that $k[(I^e)_c]$ is Cohen-Macaulay for all $e > 0$ and $c \geq ef$.*

Let $\pi : X \to Y$ be the blow-up morphism, $E = \mathcal{P}roj(\bigoplus_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1})$, and w_E its dualizing sheaf. The main result they obtain is the following general criterion:

Theorem [CH, Theorem 4.1] *Suppose that $I \subset A$ is a homogeneous ideal such that $I \not\subset \mathfrak{p}$, $\forall \mathfrak{p} \in \mathrm{Ass}\,(A)$, A is Cohen-Macaulay and X is a Cohen-Macaulay scheme. Suppose that $\pi_* \mathcal{O}_E(m) = \mathcal{I}^m/\mathcal{I}^{m+1}$ for $m \geq 0$, $R^i \pi_* \mathcal{O}_E(m)=0$ for $i > 0$ and $m \geq 0$, $R^i \pi_* w_E(m)=0$ for $i > 0$ and $m \geq 2$. Then there exists a positive integer f such that $k[(I^e)_c]$ is Cohen-Macaulay for $e > 0$ and $c \geq ef$.*

This result is applied there to the following classes of ideals:

Corollary [CH, Corollary 4.2] *Let $I \subset A$ be a homogeneous ideal such that $I \not\subset \mathfrak{p}$, $\forall \mathfrak{p} \in \mathrm{Ass}\,(A)$, A is Cohen-Macaulay and $I_{\mathfrak{p}}$ is a complete intersection ideal for any $\mathfrak{p} \in \mathrm{Proj}\ (A)$. Then there exists a positive integer f such that $k[(I^e)_c]$ is Cohen-Macaulay for $e > 0$, $c \geq ef$.*

Corollary [CH, Corollary 4.4] *Let $I \subset A$ be a homogeneous ideal such that $I \not\subset \mathfrak{p}$, $\forall \mathfrak{p} \in \mathrm{Ass}\,(A)$, A is Cohen-Macaulay and $I_{(\mathfrak{p})}$ is strongly Cohen-Macaulay with $\mu(I_{(\mathfrak{p})}) \leq \mathrm{ht}\,(\mathfrak{p})$ for any prime ideal $\mathfrak{p} \in \mathrm{Proj}\ (A)$ containing I. Then there exists a positive integer f such that $k[(I^e)_c]$ is Cohen-Macaulay for $e > 0$, $c \geq ef$.*

As a somehow unexpected by-product, the methods used to study the diagonals of a Rees algebra also allow to study the regularity of the powers of an ideal and their asymptotic properties. These problems have been previously handled by using other techniques. Let $A = k[X_1, \ldots, X_n]$ be a polynomial ring with the usual grading and let $I \subset A$ be a homogeneous ideal. I. Swanson [Swa] has shown that there exists an integer B such that $\mathrm{reg}\,(I^e) \leq Be$, $\forall e$. The problem is to make B explicit. In some particular cases, such B was already known. A. Geramita, A. Gimigliano and Y. Pitteloud [GGP] and K. Chandler [Cha] had proved that for ideals with $\dim(A/I) = 1$, $\mathrm{reg}\,(I^e) \leq \mathrm{reg}\,(I)\, e$. On the other hand, R. Sjögren [Sjo] had given another kind of bound: If I is an ideal generated by forms of degree $\leq d$ with $\dim(A/I) \leq 1$, $\mathrm{reg}\,(I^e) < (n-1)de$. Also A. Bertram, L. Ein and R. Lazarsfeld [BEL] have given a bound for the regularity of the powers of an ideal in terms of the degrees of its generators: If I is the ideal of a smooth complex subvariety X of $\mathbb{P}^{n-1}_{\mathbb{C}}$ of codimension c generated by forms of degrees $d_1 \geq d_2 \geq \ldots \geq d_r$, then

$$H^i(\mathbb{P}^{n-1}_{\mathbb{C}}, \mathcal{I}^e(k)) = 0, \ \ \forall i \geq 1, \forall k \geq ed_1 + d_2 + \ldots + d_c - (n-1).$$

Let $(A, \mathfrak{m}, k)$ be a local ring and let $I \subset A$ be an ideal. Concerning the asymptotic properties of the powers of I, a classical well known result of

M. Brodmann [Bro] says that $\operatorname{depth} A/I^j$ takes a constant asymptotic value C for $j >> 0$, and moreover $C \leq \dim A - l(I)$. This value C was determined by D. Eisenbud and C. Huneke [EH] for ideals under some restrictions: If I is an ideal of height greater than zero and $G_A(I)$ is Cohen-Macaulay, then $\inf\{\operatorname{depth} A/I^j\} = \dim A - l(I)$, and if $\operatorname{depth} A/I^s = \inf\{\operatorname{depth} A/I^j\}$, then $\operatorname{depth} A/I^{s+1} = \operatorname{depth} A/I^s$. Finally, V. Kodiyalam [Ko1] has shown that for any fixed nonnegative integer p and all sufficiently large j, the p-th Betti number $\beta_p^A(I^j) = \dim_k \operatorname{Tor}_p^A(I^j, k)$ and the p-th Bass number $\mu_A^p(I^j) = \dim_k \operatorname{Ext}_A^p(k, I^j)$ are polynomials in j of degree $\leq l(I) - 1$.

Now we are going to set and motivate the concrete problems and questions considered in this dissertation.

The restriction to Rees algebras of equigenerated ideals done by A. Simis et al. [STV] is due to the fact that in this case the Rees algebra can be endowed with a bigrading so that it becomes standard. For standard bigraded algebras one may define its biprojective scheme (see [STV], [Hy]) and there are also known results about its Hilbert polynomial (see [HHRT], [KMV]). If I is an ideal generated by forms $f_1, \ldots, f_r$ of degrees $d_1, \ldots, d_r$ respectively, the Rees algebra of I has a presentation as a quotient of $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ bigraded by setting $\deg(X_i) = (1, 0)$, $\deg(Y_j) = (d_j, 1)$ which is non standard. Our first problem will be to extend the definitions and known results on bigraded modules over standard bigraded algebras to the category of bigraded S-modules.

Several arithmetic properties of a ring such as the Cohen-Macaulayness and the Gorenstein property can be characterized by means of its local cohomology modules. This is the reason why it is interesting and useful to study when the local cohomology modules and the diagonal functor commute, case in which we may conclude that certain arithmetic properties of the Rees algebra are inherited by its diagonals. The shifts (a, b) which arise in the bigraded minimal free resolution of the Rees algebra $R_A(I)$ over the polynomial ring S play an essential role in this problem as it was seen in [CHTV]. We will study and bound these shifts by relating them to the local cohomology of the Rees algebra. After that, we will focus on the obstructions for the local cohomology modules and the diagonal functor to commute.

Once we have done all those preliminaries, our main purpose will be to study the Cohen-Macaulayness of the rings $k[(I^e)_c]$. We will consider different

questions such as the existence and the determination of the diagonals (c, e) for which $k[(I^e)_c]$ is Cohen-Macaulay, problems treated in [STV], [CHTV] and [CH]. Similarly, our next goal will be to study the Gorenstein property of the k-algebras $k[(I^e)_c]$. This has been only done in a very particular case in [CHTV].

Some of the criteria we will obtain for the Cohen-Macaulayness of the k-algebras $k[(I^e)_c]$ are in terms of the local cohomology modules of the powers of the ideal I. This will lead us to study the a-invariants of the powers of a homogeneous ideal. We will then show how the bigrading defined in the Rees algebra can be used to study the a-invariants and the asymptotic properties of the powers of an ideal.

Summarizing, the main problems we have considered in this work are:

(**1**) To extend the definitions and results about the biprojective scheme and the Hilbert polynomial of finitely generated bigraded modules defined over standard bigraded k-algebras to finitely generated bigraded S-modules, for $S = k[X_1, \dots, X_n, Y_1, \dots, Y_r]$ the polynomial ring bigraded by $\deg(X_i) = (1, 0)$, $\deg(Y_j) = (d_j, 1)$, $d_1, \dots, d_r \geq 0$.

(**2**) To relate the shifts in the bigraded minimal free resolution of any finitely generated bigraded S-module to its a-invariants.

(**3**) To study the local cohomology modules of the diagonals of any finitely generated bigraded S-module.

(**4**) To study the Cohen-Macaulay property of the rings $k[(I^e)_c]$.

(**5**) To study the Gorenstein property of the rings $k[(I^e)_c]$.

(**6**) To study the a-invariants of the powers of a homogeneous ideal.

(**7**) To study the asymptotic properties of the powers of a homogeneous ideal.

Now we are ready to describe the results obtained in this work.

In **Chapter 1** we introduce the notations and definitions we will need throughout this work. We begin the chapter by defining the category of multigraded modules over a multigraded ring, and by recalling some well-known results about multigraded local cohomology and the canonical module mainly

following M. Herrmann, E. Hyry and J. Ribbe [HHR] and S. Goto and K. Watanabe [GW1]. Then we define the multigraded a-invariants of a module and we study the relationship between these a-invariants and the shifts of its multigraded minimal free resolution. We will obtain a formula which extends [BH1, Example 3.6.15], where it was proved for Cohen-Macaulay modules in the graded case. This result will be a very useful device used all along this work. To precise it, let S be a d-dimensional $\mathbb{N}^r$-graded Cohen-Macaulay k-algebra with homogeneous maximal ideal $\mathcal{M}$ and let M be a finitely generated r-graded S-module of dimension m and depth ρ. For each $i = 0, \dots, m$, we may associate to the i-th local cohomology module of M its multigraded a_i-invariant $\mathbf{a}_i(M) = (a_i^1(M), \dots, a_i^r(M))$, where

$$a_i^j(M) = \max\{n \mid \exists \mathbf{n} = (n^1, \dots, n^r) \in \mathbb{Z}^r \ s.t. \ \underline{\mathrm{H}}^i_{\mathcal{M}}(M)_{\mathbf{n}} \neq 0, n^j = n\}$$

if $\underline{\mathrm{H}}^i_{\mathcal{M}}(M) \neq 0$ and $a_i^j(M) = -\infty$ otherwise. Notice that $\mathbf{a}_m(M)$ coincides with the usual a-invariant, and so we will denote by $\mathbf{a}(M) = (a^1(M), \dots, a^r(M)) = \mathbf{a}_m(M)$. Finally, the multigraded a_*-invariant of M is $\mathbf{a}_*(M) = (a_*^1(M), \dots, a_*^r(M))$, where $a_*^j(M) = \max_{i=0,\dots,m}\{a_i^j(M)\}$.

On the other hand, we may consider the r-graded minimal free resolution of M over S. Suppose that this resolution is finite:

$$0 \to D_l \to \dots \to D_1 \to D_0 \to M \to 0,$$

with $D_p = \bigoplus_q S(a^1_{pq}, ..., a^r_{pq})$. For every $p \in \{0, \dots, l\}$, $j \in \{1, \dots, r\}$, let us denote by

$$t_p^j(M) = \max_q\{-a^j_{pq}\},$$
$$t_*^j(M) = \max_{p,q}\{-a^j_{pq}\} = \max_p t_p^j(M),$$
$$\mathbf{t}_*(M) = (t_*^1(M), \dots, t_*^r(M)).$$

Moreover, given a permutation σ of the set $\{1, ..., r\}$, let us consider $\leq_\sigma$ the order in $\mathbb{Z}^r$ defined by: $(u_1, ..., u_r) \leq_\sigma (v_1, ..., v_r)$ iff $(u_{\sigma(1)}, ..., u_{\sigma(r)}) \leq_{lex} (v_{\sigma(1)}, ..., v_{\sigma(r)})$, where $\leq_{lex}$ is the lexicographic order. Set $\mathbf{M}_p^\sigma = \max_{\leq_\sigma}\{ (-a^1_{pq}, ..., -a^r_{pq})\}$. Then we can relate the shifts and the a-invariants of M in the following way:

Theorem 1 [Theorem 1.3.4] *For every $j = 1, \dots, r$,*

(i) $a^j_{d-p}(M) \leq t_p^j(M) + a^j(S)$, *for* $p = d - m, \dots, d - \rho$.

(ii) *Assume that for some* p *there exists* σ *s.t.* $\sigma(1) = j$ *and* $M_p^\sigma >_\sigma M_{p+1}^\sigma$. *Then* $a_{d-p}^j(M) = t_p^j(M) + a^j(S)$.

(iii) $a_*^j(M) = t_*^j(M) + a^j(S)$. *That is,* $\mathbf{a}_*(M) = \mathbf{t}_*(M) + \mathbf{a}(S)$.

After that, we extend the definition and some of the results about the multiprojective scheme associated to a standard r-graded ring given by E. Hyry [Hy] and M. Herrmann et al. [HHRT] to rings endowed with a more general grading, which will also include the Rees algebra of a homogeneous ideal. Let S be a noetherian $\mathbb{N}^r$-graded ring generated over S_0 by homogeneous elements $x_{11}, \ldots, x_{1k_1}, \ldots, x_{r1}, \ldots, x_{rk_r}$ in degrees $\deg(x_{ij}) = (d_{ij}^1, \ldots, d_{ij}^{i-1}, 1, 0, \ldots, 0)$, with $d_{ij}^l \geq 0$. For every $j = 1, \ldots, r$, let I_j be the ideal of S generated by the homogeneous components of S of degree $\mathbf{n} = (n_1, \ldots, n_r)$ such that $n_j > 0, n_{j+1} = \ldots = n_r = 0$. The irrelevant ideal of S is $S_+ = I_1 \cdots I_r$. We may associate to S the r-projective scheme $\text{Proj}^{\,r}(S)$ which as a set contains all the homogeneous prime ideals $P \subset S$ such that $S_+ \not\subset P$. The relevant dimension of S is

$$\text{rel.dim}\, S = \begin{cases} r-1 & \text{if Proj}^{\,r}(S) = \emptyset \\ \max\{\dim S/P \mid P \in \text{Proj}^{\,r}(S)\} & \text{if Proj}^{\,r}(S) \neq \emptyset \end{cases}.$$

It can be proved that $\dim \text{Proj}^{\,r}(S) = \text{rel.dim}\, S - r$ by arguing as in [Hy, Lemma 1.2] where the standard r-graded case was considered. This result jointly with the isomorphism of schemes $\text{Proj}^{\,r}(S) \cong \text{Proj}\,(S_\Delta)$ that we have for certain diagonals allows to compute the dimension of S_Δ whenever S_0 is artinian, by extending [STV, Proposition 2.3] where this dimension was determined for the $(1,1)$-diagonal of a standard bigraded k-algebra by different methods.

Finally, we extend to the category of r-graded modules defined over the r-graded k-algebras introduced before the basic results concerning Hilbert functions and Hilbert polynomials. Some of them have been established in the standard r-graded case in [HHRT] and [KMV].

In **Chapter 2** we are concerned with the diagonal functor in the category of bigraded S-modules, where S is the polynomial ring $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ bigraded by setting $\deg X_i = (1,0)$, $\deg Y_j = (d_j, 1)$, $d_1, \ldots, d_r \geq 0$. In the first section, we compare the local cohomology modules of a finitely generated bigraded S-module L with the local cohomology modules of its diagonals. In particular, we will prove the main results in [CHTV] by

a different and somewhat easier approach. In addition, this approach will provide more detailed information about several problems related to the behaviour of the local cohomology when taking diagonals. Set $d = \max\{d_1, \dots, d_r\}$, and let $\Delta = (c, e)$ be a diagonal with $c \geq de + 1$. Let us consider the following subalgebras of S: $S_1 = k[X_1, \dots, X_n]$, $S_2 = k[Y_1, \dots, Y_r]$, with homogeneous maximal ideals $\mathfrak{m}_1 = (X_1, ..., X_n)$ and $\mathfrak{m}_2 = (Y_1, \dots, Y_r)$. Let $\mathcal{M}_1$, $\mathcal{M}_2$ be the ideals of S generated by $\mathfrak{m}_1$, $\mathfrak{m}_2$ respectively, and let $\mathcal{M}$ be the homogeneous maximal ideal of S. Then:

Proposition 2 [Proposition 2.1.3] *Let L be a finitely generated bigraded S-module. There exists a natural exact sequence*

$$... \to H^q_{\mathcal{M}}(L)_\Delta \to H^q_{\mathcal{M}_1}(L)_\Delta \oplus H^q_{\mathcal{M}_2}(L)_\Delta \to H^q_{\mathcal{M}_\Delta}(L_\Delta) \xrightarrow{\varphi^q_L} H^{q+1}_{\mathcal{M}}(L)_\Delta \to ...$$

In the rest of the section, we study the obstructions for φ^q_L to be an isomorphism. Firstly, we relate this question to the vanishing of the local cohomology with respect to $\mathcal{M}_1$ and $\mathcal{M}_2$ of the modules $S(a, b)$ which arise in the bigraded minimal free resolution of L over S. This allows us as said to recover the main results in [CHTV]. After that, we study the vanishing of the local cohomology modules of L with respect to $\mathcal{M}_1$ and $\mathcal{M}_2$ by themselves.

In Section 2.2 we will focus on standard bigraded k-algebras. Given a standard bigraded k-algebra R, let us consider the graded subalgebras $\mathcal{R}_1 = \bigoplus_{i\in\mathbb{N}} R_{(i,0)}$, $\mathcal{R}_2 = \bigoplus_{j\in\mathbb{N}} R_{(0,j)}$. By using Theorem 1, we obtain a characterization for R to have a good resolution in terms of the a_*-invariants of $\mathcal{R}_1$ and $\mathcal{R}_2$ which, in particular, provides a criterion for the Cohen-Macaulay property of its diagonals. We also find necessary and sufficient conditions on the local cohomology of $\mathcal{R}_1$ and $\mathcal{R}_2$ for the existence of Cohen-Macaulay diagonals of a Cohen-Macaulay standard bigraded k-algebra R. This result extends [CHTV, Corollary 3.12].

Proposition 3 [Proposition 2.2.7] *Let R be a standard bigraded Cohen-Macaulay k-algebra of relevant dimension δ. There exists Δ such that R_Δ is Cohen-Macaulay if and only if $H^q_{\mathfrak{m}_1}(\mathcal{R}_1)_0 = H^q_{\mathfrak{m}_2}(\mathcal{R}_2)_0 = 0$ for any $q < \delta - 1$.*

Now let us consider a standard bigraded ring R defined over a local ring with $a^1(R), a^2(R) < 0$. In [Hy, Theorem 2.5] it is shown that if R is Cohen-Macaulay then the $\Delta = (1, 1)$-diagonal of R has also this property. This result can be extended to any diagonal of a standard bigraded k-algebra:

Proposition 4 [Proposition 2.2.6] *Let R be a standard bigraded Cohen-Macaulay k-algebra with $a^1(R), a^2(R) < 0$. Then R_Δ is Cohen-Macaulay for any diagonal Δ.*

At the end of the chapter, we apply the results about bigraded k-algebras to the Rees algebra of a homogeneous ideal. Let A be a noetherian graded k-algebra generated in degree 1 of dimension $\overline{n}$ and let $\mathfrak{m}$ be the homogeneous maximal ideal of A. Given a homogeneous ideal I of A, the Rees algebra $R = R_A(I)$ of I is bigraded by $R_A(I)_{(i,j)} = (I^j)_i$. If I is generated in degree $\leq d$, for any diagonal $\Delta = (c, e)$ with $c \geq de + 1$ we have:

$$R_A(I)_\Delta = k[(I^e)_c].$$

The diagonals $k[(I^e)_c]$ are graded k-algebras of dimension $\overline{n}$ if no associated prime of A contains I. In the sequel we will always assume such hypothesis. We can relate the local cohomology modules of the k-algebras $k[(I^e)_c]$ and those of the powers of I. Denoting by m the homogeneous maximal ideal of $k[(I^e)_c]$, we have:

Proposition 5 [Corollary 2.3.5] *For any $c \geq de + 1$, $e > a^2_*(R)$, $s > 0$, we have isomorphisms*

$$H^q_m(k[(I^e)_c])_s \cong H^q_{\mathfrak{m}}(I^{es})_{cs}, \forall q \geq 0.$$

In the particular case where $A = k[X_1, \ldots, X_n]$, A. Conca et al. [CHTV] conjectured that if the Rees algebra of a homogeneous ideal I of A is Cohen-Macaulay, then there exists a Cohen-Macaulay diagonal. The results proved for standard bigraded k-algebras provide an affirmative answer for equigenerated homogeneous ideals. In fact, we can give a full answer to this conjecture.

Theorem 6 [Theorem 2.3.12] *Let I be a homogeneous ideal of the polynomial ring $A = k[X_1, \ldots, X_n]$. If $R_A(I)$ is a Cohen-Macaulay ring, then $R_A(I)$ has a good resolution. In particular, $k[(I^e)_c]$ is Cohen-Macaulay for $c \gg e \gg 0$.*

Furthermore, we obtain sufficient and necessary conditions on the ring A for the existence of Cohen-Macaulay diagonals of a Rees algebra $R_A(I)$ with this property. Namely,

Theorem 7 [Theorem 2.3.13] *If $R_A(I)$ is Cohen-Macaulay, then the following are equivalent:*

(i) *There exist* c, e *such that* $k[(I^e)_c]$ *is Cohen-Macaulay.*

(ii) $H^i_{\mathfrak{m}}(A)_0 = 0$ *for* $i < \overline{n}$.

In **Chapter 3** we study in detail the Cohen-Macaulay property of the rings $k[(I^e)_c]$. We consider the problem of the existence of Cohen-Macaulay diagonals of the Rees algebra. Once studied this problem, we will try to determine the diagonals with this property. The following isomorphisms will play an important role:

Proposition 8 [Proposition 3.1.2] *Let* X *be the blow-up of* Proj (A) *along* $\mathcal{I} = \widetilde{I}$, *where* I *is a homogeneous ideal of* A *generated by forms of degree* $\leq d$. *For any* $c \geq de + 1$, *there are isomorphisms of schemes*

$$X \cong \text{Proj}\ ^2(R_A(I)) \cong \text{Proj}\ (k[(I^e)_c]).$$

First of all, these isomorphisms will be used to give a criterion for the existence of diagonals $k[(I^e)_c]$ which are generalized Cohen-Macaulay modules, thereby solving a conjecture of [CHTV].

Proposition 9 [Proposition 3.2.6] *The following are equivalent:*

(i) $H^i_{\mathcal{M}}(R_A(I))_{(p,q)}$=0 *for* $i < \overline{n} + 1$, $p \ll q \ll 0$.

(ii) $k[(I^e)_c]$ *is a generalized Cohen-Macaulay module for* $c \gg e \gg 0$.

(iii) *There exist* c, e *such that* $k[(I^e)_c]$ *is generalized Cohen-Macaulay.*

(iv) $k[(I^e)_c]$ *is a Buchsbaum ring for* $c \gg e \gg 0$.

(v) *There exist* c, e *such that* $k[(I^e)_c]$ *is a Buchsbaum ring.*

(vi) *There exist* q_0, t *such that* $H^i_{\mathcal{M}}(R_A(I))_{(p,q)} = 0$ *for* $i < \overline{n}+1$, $q < q_0$ *and* $p < dq + t$.

After that, we use Proposition 8 to give necessary and sufficient conditions for a Rees algebra to have Cohen-Macaulay diagonals. Namely,

Theorem 10 [Theorem 3.2.3, Corollary 3.2.5] *The following are equivalent:*

(i) *There exist* c, e *such that* $k[(I^e)_c]$ *is a Cohen-Macaulay ring.*

(ii) *(1)* *There exist* $q_0, t \in \mathbb{Z}$ *such that* $H^i_{\mathcal{M}}(R_A(I))_{(p,q)} = 0$ *for all* $i < \overline{n}+1$, $q < q_0$ *and* $p < dq + t$.

(2) $H^i_{R_A(I)_+}(R_A(I))_{(0,0)} = 0$ *for all* $i < \overline{n}$.

(iii) *(1)* X *is an equidimensional Cohen-Macaulay scheme.*

(2) $\Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) = k$, $H^i(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) = 0$ *for* $0 < i < \overline{n} - 1$.

In this case, $k[(I^e)_c]$ *is a Cohen-Macaulay ring for* $c \gg e \gg 0$.

By using this theorem, we can exhibit some general situations in which we can ensure the existence of Cohen-Macaulay coordinate rings for X. For instance,

Proposition 11 [Proposition 3.3.3] *Let* X *be the blow-up of* $\mathbb{P}^{n-1}_k$ *along a closed subscheme, where* k *has* $\operatorname{char} k = 0$. *Assume that* X *is smooth or with rational singularities. Then* X *is arithmetically Cohen-Macaulay.*

Our next goal in the chapter will be to determine the Cohen-Macaulay diagonals once we know its existence. This is a difficult problem which has been completely solved only for complete intersection ideals in the polynomial ring [CHTV, Theorem 4.6]. For equigenerated ideals, we can give a criterion for the Cohen-Macaulayness of a diagonal in terms of the local cohomology modules of the powers of the ideal by just assuming that the Rees algebra is Cohen-Macaulay. Namely,

Proposition 12 [Proposition 3.4.1] *Let* $I \subset A$ *be an ideal generated by forms of degree* d *whose Rees algebra is Cohen-Macaulay. For any* $c \geq de+1$, $k[(I^e)_c]$ *is Cohen-Macaulay if and only if*

(i) $H^i_{\mathfrak{m}}(A)_0 = 0$ *for* $i < \overline{n}$.

(ii) $H^i_{\mathfrak{m}}(I^{es})_{cs} = 0$ *for* $i < \overline{n}$, $s > 0$.

For arbitrary homogeneous ideals, we can also prove a criterion for the Cohen-Macaulayness of a diagonal by means of the local cohomology of the powers of the ideal and the local cohomology of the graded pieces of the canonical module of the Rees algebra. Let us denote by $K = K_{R_A(I)} = \bigoplus_{(i,j)} K_{(i,j)}$ the canonical module of the Rees algebra, and for each $e \in \mathbb{Z}$, let us consider the graded A-module $K^e = \bigoplus_i K_{(i,e)}$. Then we have:

Theorem 13 [Theorem 3.4.3] *Let* I *be a homogeneous ideal of* A *generated by forms of degree* $\leq d$ *whose Rees algebra is Cohen-Macaulay. For any* $c \geq de+1$, $k[(I^e)_c]$ *is Cohen-Macaulay if and only if*

(i) $H^i_{\mathfrak{m}}(A)_0 = 0$ *for* $i < \overline{n}$.

(ii) $H^i_{\mathfrak{m}}(I^{es})_{cs} = 0$ *for* $i < \overline{n}$, $s > 0$.

(iii) $H^{\overline{n}-i+1}_{\mathfrak{m}}(K^{es})_{cs} = 0$ *for* $1 \leq i < \overline{n}$, $s > 0$.

If the form ring is quasi-Gorenstein we can express the criterion above only in terms of the local cohomology of the powers of the ideal.

Theorem 14 [Corollary 3.4.4] *Let* I *be a homogeneous ideal of* A *generated by forms of degree* $\leq d$. *Assume that* $R_A(I)$ *is Cohen-Macaulay,* $G_A(I)$ *is quasi-Gorenstein. Set* $a = -a^2(G_A(I))$, $b = -a(A)$. *For any* $c \geq de+1$, $k[(I^e)_c]$ *is Cohen-Macaulay if and only if*

(i) $H^i_{\mathfrak{m}}(A)_0 = 0$ *for* $i < \overline{n}$.

(ii) $H^i_{\mathfrak{m}}(I^{es})_{cs} = 0$ *for* $i < \overline{n}$, $s > 0$.

(iii) $H^i_{\mathfrak{m}}(I^{es-a+1})_{cs-b} = 0$ *for* $1 < i \leq \overline{n}$, $s > 0$.

We can use Theorem 14 to determine exactly the Cohen-Macaulay diagonals of the Rees algebra of a complete intersection ideal in any Cohen-Macaulay ring. In particular, we get a new proof of [CHTV, Theorem 4.6] where the case $A = k[X_1, \ldots, X_n]$ was studied.

These criteria will be also applied in the Chapter 5, once we have studied in detail the local cohomology modules of the powers of several families of ideals, such as equimultiple ideals or strongly Cohen-Macaulay ideals.

Furthermore, the results and methods used up to now allow us to show the behaviour of the a_*-invariant of the powers of a homogeneous ideal. The following statement has been obtained independently by S.D. Cutkosky, J. Herzog and N. V. Trung [CHT] and V. Kodiyalam [Ko2] by different methods.

Theorem 15 [Theorem 3.4.6] *Let* L *be a finitely generated bigraded* S*-module. Then there exists* α *such that*

$$a_*(L^e) \leq de + \alpha,\ \forall e.$$

After that, we use the bound on the shifts of the bigraded minimal free resolution of the Rees algebra obtained in Theorem 1 to determine a family of Cohen-Macaulay diagonals of a Cohen-Macaulay Rees algebra.

Theorem 16 [Theorem 3.4.12] *Let I be a homogeneous ideal of A generated by r forms of degree $d_1 \leq \ldots \leq d_r = d$. Assume that $H^i_{\mathfrak{m}}(A)_0 = 0$ for $i < \overline{n}$. Set $u = \sum_{j=1}^{r} d_j$. If the Rees algebra is Cohen-Macaulay, then*

(i) $k[(I^e)_c]$ *is Cohen-Macaulay for* $c > \max\{d(e-1) + u + a(A), d(e-1) + u - d_1(r-1)\}$.

(ii) If I is generated by forms in degree d, $k[(I^e)_c]$ *is Cohen-Macaulay for* $c > d(e-1+l(I)) + a(A)$.

Our results can be also applied to study the embedddings of the blow-up of a projective space along an ideal I of fat points via the linear systems $(I^e)_c$ whenever these linear systems are very ample, slightly extending [GGP, Theorem 2.4] where only the linear systems I_c were considered.

Theorem 17 [Theorem 3.4.15] *Let $I \subset A = k[X_1, \ldots, X_n]$ be an ideal of fat points, with k a field of characteristic zero. Then*

(i) $k[(I^e)_c]$ *is Cohen-Macaulay if and only if* $H^i_{\mathfrak{m}}(I^{es})_{cs} = 0$ *for* $s > 0$, $i < n$.

(ii) For $c > \operatorname{reg}(I)e$, $k[(I^e)_c]$ *is Cohen-Macaulay with* $a(k[(I^e)_c]) < 0$. *In particular,* $\operatorname{reg}(k[(I^e)_c]) < n - 1$.

The chapter finishes by studying sufficient conditions for the existence of a positive integer f such that $k[(I^e)_c]$ is a Cohen-Macaulay ring for all $c \geq ef$ and $e > 0$, a question that has been treated by S.D. Cutkosky and J. Herzog. Our main result, which improves [CH, Corollaries 4.2, 4.3 and 4.4], is the following:

Theorem 18 [Theorem 3.5.3] *Let I be a homogeneous ideal of an equidimensional k-algebra A such that $R_{A_{\mathfrak{p}}}(I_{\mathfrak{p}})$ is Cohen-Macaulay for any prime ideal $\mathfrak{p} \in \operatorname{Proj}(A)$. Assume that $H^i_{\mathfrak{m}}(A)_0 = 0$ for $i < \overline{n}$. Then there exists an integer α such that $k[(I^e)_c]$ is Cohen-Macaulay for all $c \geq de + \alpha$ and $e > 0$.*

The aim of **Chapter 4** is to study the Gorenstein property of the k-algebras $k[(I^e)_c]$. About the Cohen-Macaulay property, we have already proved that if there exists a Cohen-Macaulay diagonal then there are infinitely many with this property. We show that the behaviour of the Gorenstein property is totally different. For instance, by considering the polynomial ring $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ with $\deg X_i = (1, 0)$, $\deg Y_j = (d_j, 1)$,

$d_1, \ldots, d_r \geq 0$, we have that S_Δ is Cohen-Macaulay for any diagonal Δ but there is just a finite set of Gorenstein diagonals.

Proposition 19 [Proposition 4.1.1] *S_Δ is Gorenstein if and only if $\frac{r}{e} = \frac{n+u}{c} = l \in \mathbb{Z}$. Then $a(S_\Delta) = -l$.*

To determine the rings $k[(I^e)_c]$ which are Gorenstein, we will compare the canonical module of the Rees algebra with the canonical module of each diagonal. For complete intersection ideals of the polynomial ring, it was proved in [CHTV, Proposition 4.5] that the canonical module and the diagonal functor commute. This result can be extended to more general situations.

Proposition 20 [Proposition 4.1.4 and Remark 4.1.5] *Let $A = k[X_1, \ldots, X_n]$ be the polynomial ring, $n \geq 2$, and let I be a homogeneous ideal of A with $\mu(I) \geq 2$.*

(i) If $\mu(I) \leq n$, $K_{R_\Delta} \cong (K_R)_\Delta$.

(ii) If I is equigenerated and R is Cohen-Macaulay, $K_{R_\Delta} \cong (K_R)_\Delta$.

Although this isomorphism can be extended to a more general class of rings, we will restrict our attention to the above two cases. This will suffice to study the rational surfaces obtained by blowing-up the projective plane at a set of points.

Next we study the behaviour of the Gorenstein property of the Rees algebra when we take diagonals. If the Rees algebra is Gorenstein then the form ring is also Gorenstein. Under this assumption on the form ring, which is less restrictive, we can determine exactly for which c, e the algebra $k[(I^e)_c]$ is quasi-Gorenstein. Namely,

Theorem 21 [Theorem 4.1.9] *Let $I \subset A = k[X_1, \ldots, X_n]$ be a homogeneous ideal with $1 < \operatorname{ht}(I) < n$ whose form ring $G_A(I)$ is Gorenstein. Set $a = -a^2(G_A(I))$. Then $k[(I^e)_c]$ is a quasi-Gorenstein ring if and only if $\frac{n}{c} = \frac{a-1}{e} = l_0 \in \mathbb{Z}$. In this case, $a(k[(I^e)_c]) = -l_0$.*

For homogeneous non principal ideals I of height 1, the ring $k[(I^e)_c]$ is never Gorenstein. If I has height n, then the diagonals determined in the theorem are always Gorenstein, but the converse is not true. As a corollary of this result we can solve the problem of determining completely the Gorenstein diagonals for complete intersection ideals or determinantal ideals generated by the maximal minors of a generic matrix.

Corollary 22 [Corollary 4.1.12] *Let $I \subset A = k[X_1, \ldots, X_n]$ be a complete intersection ideal minimally generated by r forms of degree $d_1 \leq \ldots \leq d_r = d$, with $r < n$. For any $c \geq de + 1$, $k[(I^e)_c]$ is a Gorenstein ring if and only if $\frac{n}{c} = \frac{r-1}{e} = l_0 \in \mathbb{Z}$. In this case, $a(k[(I^e)_c]) = -l_0$.*

Corollary 23 [Example 4.1.13] *Let $\mathbf{X} = (X_{ij})$ denote a matrix of indeterminates, with $1 \leq i \leq n, 1 \leq j \leq m$ and $m \leq n$. Let $I \subset A = k[\mathbf{X}]$ denote the ideal generated by the maximal minors of $\mathbf{X}$, where k is a field. Then:*

(i) If $m < n$, then $k[(I^e)_c]$ is Gorenstein if and only if $\frac{nm}{c} = \frac{n-m}{e} \in \mathbb{Z}$.

(ii) If $m = n$, then $\Delta = (n(n+1), 1)$ is the only Gorenstein diagonal.

We have shown that if the form ring is Gorenstein there is just a finite set of Gorenstein diagonals. This fact also holds under the general assumptions of the chapter. Namely,

Proposition 24 [Proposition 4.2.1] *There is a finite set of diagonals $\Delta = (c, e)$ such that $k[(I^e)_c]$ is quasi-Gorenstein.*

If the Rees algebra is Cohen-Macaulay, then we can bound the diagonals $\Delta = (c, e)$ for which $k[(I^e)_c]$ is Gorenstein.

Proposition 25 [Proposition 4.2.2] *Assume that $\mathrm{ht}\,(I) \geq 2$ and $R_A(I)$ is Cohen-Macaulay. Let $a = -a^2(G_A(I))$. If $k[(I^e)_c]$ is quasi-Gorenstein, then $e \leq a - 1$ and $c \leq n$. Moreover, if $\dim(A/I) > 0$ then $\lceil \frac{a}{e} \rceil - 1 = \frac{n}{c} = l \in \mathbb{Z}$. In particular, if $a = 1$ there are no diagonals (c, e) such that $k[(I^e)_c]$ is quasi-Gorenstein.*

Finally, we show that in some cases the existence of a diagonal (c, e) such that $k[(I^e)_c]$ is quasi-Gorenstein forces the form ring to be Gorenstein. It may be seen as a converse of Theorem 21 for those cases.

Theorem 26 [Theorem 4.2.3] *Assume that $R_A(I)$ is Cohen-Macaulay, $\mathrm{ht}\,(I) \geq 2$, $l(I) < n$ and I is equigenerated. If there exists a diagonal (c, e) such that $k[(I^e)_c]$ is quasi-Gorenstein then $G_A(I)$ is Gorenstein.*

We finish the chapter by applying the previous results to recover the fact that the Del Pezzo sestic surface in $\mathbb{P}^6$ is the only Room surface which is Gorenstein.

In **Chapter 5** we study the a-invariant and the regularity of any finitely generated bigraded S-module L, for $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ the polyno-

mial ring with $\deg X_i = (1,0)$, $\deg Y_j = (0,1)$. This class of modules includes for instance any standard bigraded k-algebra R.

Given a finitely generated bigraded S-module L, let us consider the bigraded minimal free resolution of L over S

$$0 \to D_l \to \ldots \to D_0 \to L \to 0,$$

with $D_p = \bigoplus_{(a,b)\in\Omega_p} S(a,b)$. The bigraded regularity of L is $\mathbf{reg}(L) = (\operatorname{reg}_1 L, \operatorname{reg}_2 L)$, where

$$\operatorname{reg}_1 L = \max_p \{-a-p \mid (a,b) \in \Omega_p\},$$
$$\operatorname{reg}_2 L = \max_p \{-b-p \mid (a,b) \in \Omega_p\}.$$

For each $e \in \mathbb{Z}$, we may define the graded S_1-module $L^e = \bigoplus_{i\in\mathbb{Z}} L_{(i,e)}$ and the graded S_2-module $L_e = \bigoplus_{j\in\mathbb{Z}} L_{(e,j)}$. Our first result gives a new description of the a_*-invariant $\mathbf{a}_*(L)$ of L and the regularity $\mathbf{reg}(L)$ of L in terms of the a_*-invariants and the regularities of the graded modules L^e and L_e. Namely,

Theorem 27 [Theorem 5.1.1, Theorem 5.1.2] *Let L be a finitely generated bigraded S-module. Then:*

(i) $a_*^1(L) = \max_e\{a_*(L^e)\} = \max_e\{a_*(L^e) \mid e \le a_*^2(L) + r\}$.

(ii) $a_*^2(L) = \max_e\{a_*(L_e)\} = \max_e\{a_*(L_e) \mid e \le a_*^1(L) + n\}$.

(iii) $\operatorname{reg}_1 L = \max_e\{\operatorname{reg}(L^e)\} = \max_e\{\operatorname{reg}(L^e) \mid e \le a_*^2(L) + r\}$.

(iv) $\operatorname{reg}_2 L = \max_e\{\operatorname{reg}(L_e)\} = \max_e\{\operatorname{reg}(L_e) \mid e \le a_*^1(L) + n\}$.

This result will be used to study the a_*-invariant and the regularity of the powers of a homogeneous ideal I in the polynomial ring $A = k[X_1,\ldots,X_n]$. According to Theorem 15, there exists an integer α such that $a_*(I^e) \le de + \alpha$, $\forall e$. The first aim is to determine such an α explicitly, and this will be done for any equigenerated ideal by means of a suitable a-invariant of the Rees algebra. For a homogeneous ideal I, we will denote by R, G and F the Rees algebra of I, its form ring and the fiber cone respectively. If I is an ideal generated by forms in degree d, let us denote by R^φ the Rees algebra endowed with the bigrading $[R^\varphi]_{(i,j)} = (I^j)_{i+dj}$. Then we have

Theorem 28 [Theorem 5.2.1] *Let I be a homogeneous ideal of A generated by forms in degree d. Set $l = l(I)$. Then*

(i) $a_*^1(R^\varphi) = \max_e\{\, a_*(I^e) - de\} = \max\{\, a_*(I^e) - de \mid e \leq a_*^2(R) + l\}$.

(ii) $\operatorname{reg}_1(R^\varphi) = \max_e\{\operatorname{reg}(I^e) - de\} = \max\{\operatorname{reg}(I^e) - de \mid e \leq a_*^2(R) + l\}$.

Therefore, we need to study $a_*^1(R^\varphi)$ to get concrete bounds for the a_*-invariant of the powers of several families of ideals. If the Rees algebra is Cohen-Macaulay we have

Proposition 29 [Proposition 5.2.5] *Let I be a homogeneous ideal generated by forms in degree d whose Rees algebra is Cohen-Macaulay. Set $l = l(I)$. Then*

$$-n + d(-a^2(G) - 1) \leq \max_{e \geq 0}\{a_*(I^e) - de\} \leq -n + d(l - 1).$$

The a_*-invariants of the powers of a complete intersection ideal are well-known, and in this case the inequalities above are sharp. Next we compute explicitly $a_*^1(R^\varphi) = \max_{e\geq 0}\{a_*(I^e) - de\}$ for other families of ideals. First we consider equimultiple ideals.

Proposition 30 [Proposition 5.2.8] *Let I be an equimultiple ideal equigenerated in degree d and set $h = \operatorname{ht}(I)$. If the Rees algebra is Cohen-Macaulay,*

(i) $a(I^e/I^{e+1}) = de + a(A/I)$. *In particular,* $a^1(G^\varphi) = a(A/I)$.

(ii) $a_{n-h+1}(I^e) = d(e-1) + a(A/I)$. *In particular,* $a^1(R^\varphi) = a(A/I) - d$.

For ideals whose form ring is Gorenstein we can also compute explicitly $\max_{e\geq 0}\{a_*(I^e) - de\}$, and then we get that the lower bound given by Proposition 29 is sharp.

Proposition 31 [Proposition 5.2.9] *Let I be a homogeneous ideal equigenerated in degree d whose form ring is Gorenstein. Set $l = l(I)$. Then*

(i) $\max_{e\geq 0}\{a_*(I^e) - de\} = d(-a^2(G) - 1) - n$.

(ii) *For $e > a^2(G) - a(F)$,* $\operatorname{depth}(A/I^e) = n - l$ *and* $a_*(I^e) = a_{n-l}(A/I^e) = d(e - a^2(G) - 1) - n$.

For instance, we may apply this result to determinantal ideals generated by the maximal minors of a generic matrix as well as to strongly Cohen-Macaulay ideals satisfying condition $(\mathcal{F}_1)$.

The computation of the a_*-invariants of the powers of these families of ideals is then applied to determine the Cohen-Macaulay diagonals of a Rees algebra. For equimultiple ideals, we have

Proposition 32 [Proposition 5.2.20] *Let I be an equimultiple ideal generated in degree d whose Rees algebra is Cohen-Macaulay. For any $c \geq de+1$, $k[(I^e)_c]$ is Cohen-Macaulay if and only if $c > d(e-1) + a(A/I)$.*

For strongly Cohen-Macaulay ideals, we have

Proposition 33 [Proposition 5.2.21] *Let I be a strongly Cohen-Macaulay ideal such that $\mu(I_\mathfrak{p}) \leq \mathrm{ht}\,(\mathfrak{p})$ for any prime ideal $\mathfrak{p} \supseteq I$. Assume that I is minimally generated by forms of degree $d = d_1 \geq \ldots \geq d_r$, and let $h = \mathrm{ht}\,(I)$. For $c > d(e-1) + d_1 + \ldots + d_h - n$, $k[(I^e)_c]$ is Cohen-Macaulay.*

If the Rees algebra is Cohen-Macaulay, we have proved the existence of an integer α such that $k[(I^e)_c]$ is a Cohen-Macaulay ring for any $c > de + \alpha$ and $e > 0$ by Theorem 16. For equigenerated ideals we had $\alpha = d(l-1)$ as an upper bound. We can determine the best α.

Proposition 34 [Proposition 5.2.15, Corollary 5.2.16] *Let I be an ideal in the polynomial ring $A = k[X_1, \ldots, X_n]$ generated by forms in degree d whose Rees algebra is Cohen-Macaulay. Set $l = l(I)$. For $\alpha \geq 0$, the following are equivalent*

(i) $k[(I^e)_c]$ *is CM for* $c > de + \alpha$.

(ii) $a_i(I^e) \leq de + \alpha$, $\forall i$, $\forall e$.

(iii) $a_i(I^e) \leq de + \alpha$, $\forall i$, $\forall e \leq l-1$.

(iv) $H^{n+1}_{\mathcal{M}}(R_A(I))_{(p,q)} = 0$, $\forall p > dq + \alpha$, *that is,* $\alpha \geq a^1(R^\varphi)$.

(v) The minimal bigraded free resolution of $R_A(I)$ is good for any diagonal $\Delta = (c, e)$ such that $c > de + \alpha$.

If the form ring is Gorenstein, these conditions are equivalent to

(vi) $\alpha \geq d(-a^2(G) - 1) - n$.

Up to now, we have used Theorem 27 to bound the a_*-invariants of the powers of an ideal, which has been applied to study the Cohen-Macaulayness of the diagonals. In the last section, we use this theorem to prove a bigraded

version of the Bayer-Stillman theorem which characterizes the bigraded regularity of a homogeneous ideal of S by means of generic homogeneous forms. Next, similarly to the graded case, we define the generic initial ideal $\mathbf{gin}I$ of a homogeneous ideal I of S and we establish its basic properties. In particular, we may use the Bayer-Stillman theorem to compute the regularity of a Borel-fix ideal in S when k has characteristic zero. For $j = 1, 2$, let us denote by $\delta_j(I)$ the maximum of the j-th component of the degrees in a minimal homogeneous system of generators of I. Then we have

Proposition 35 [Proposition 5.3.10] *Let $I \subset S$ be a Borel-fix ideal. If $\operatorname{char} k = 0$, then*

$$\operatorname{reg}_1(I) = \delta_1(I),$$
$$\operatorname{reg}_2(I) = \delta_2(I).$$

This result has been also proved by A. Aramova et al. [ACD] by different methods. In the graded case, D. Bayer and M. Stillman [BaSt] also proved the existence of an order in the polynomial ring $A = k[X_1, \ldots, X_n]$ (the reverse lexicographic order) such that $\operatorname{reg} I = \operatorname{reg}(\operatorname{gin} I)$ for any homogeneous ideal I of A. We finish the chapter by showing that the analogous bigraded result does not hold because we can find a homogeneous ideal I of S such that for any order $\mathbf{reg}(I) \neq \mathbf{reg}(\mathbf{gin}I)$.

In **Chapter 6** we study the asymptotic properties of the powers of a homogeneous ideal I in the polynomial ring $A = k[X_1, \ldots, X_n]$. We will show how the bigraded structure of the Rees algebra provides information about the Hilbert polynomials, the Hilbert series and the graded minimal free resolutions of the powers of I. This grading of the Rees algebra will be also useful to study the mixed multiplicities of the Rees algebra and the form ring of an equigenerated ideal.

Theorem 36 [Theorem 6.1.1] *Let I be a homogeneous ideal of A. Set $c = a_*^2(R_A(I))$, $h = \operatorname{ht}(I)$. Then there are polynomials $e_0(j), \ldots, e_{n-h-1}(j)$ with integer values such that for all $j \geq c+1$*

$$P_{A/I^j}(s) = \sum_{k=0}^{n-h-1} (-1)^{n-h-1-k} e_{n-h-1-k}(j) \binom{s+k}{k}.$$

Furthermore, $\deg e_{n-h-1-k}(j) \leq n-k-1$ *for all k.*

In particular, this result says that a finite set of Hilbert polynomials of the powers of an ideal allows to compute the Hilbert polynomials of its Rees algebra and its form ring, without needing an explicit presentation of these bigraded algebras. For equigenerated ideals, we may also compute the multiplicities of their Rees algebras and form rings.

Corollary 37 [Corollary 6.1.8] *Let I be a homogeneous ideal in A. Let $c = a^2_*(R_A(I))$, $h = \mathrm{ht}\,(I)$. Then the Hilbert polynomials of I^j for $c+1 \leq j \leq c+n$ determine*

(i) The polynomials $e_{n-h-1-k}(j)$ for $k = 0, \ldots, n-h-1$.

(ii) The Hilbert polynomials of A/I^j for $j > c+n$.

(iii) The Hilbert polynomial of $R_A(I)$ and the Hilbert polynomial of $G_A(I)$.

(iv) If I is equigenerated and not $\mathfrak{m}$-primary, the mixed multiplicities of $R_A(I)$ and $G_A(I)$.

A similar result can be proved for the Hilbert series of the powers of a homogeneous ideal. Namely,

Proposition 38 [Theorem 6.2.1, Proposition 6.2.7] *Let I be a homogeneous ideal. Set $r = \mu(I)$, $l = l(I)$, $c = a^2_*(R_A(I))$. Then:*

(i) The Hilbert series of I^j for $j \leq c+r$ determine the Hilbert series of I^j for $j > c+r$.

(ii) If I is an equigenerated ideal, the Hilbert series of I^j for $c+1 \leq j \leq c+l$ determine the Hilbert series of I^j for $j > c+l$.

Next we study the behaviour of the projective dimension of the powers of an ideal. As a by-product, we recover the classic result of M. P. Brodmann [Bro] which says that the depth of the powers of an ideal becomes constant asymptotically, and a result of D. Eisenbud and C. Huneke [EH] which precises this asymptotic value under some restrictions. Moreover, for ideals whose form ring is Gorenstein we may determine exactly the powers of the ideal for which the projective dimension takes the asymptotic value. Namely,

Proposition 39 [Proposition 6.3.2] *Let I be a homogeneous ideal in A and set $l = l(I)$. If G is Gorenstein,* $\mathrm{proj.dim}_A(I^j) \leq l-1$ *for all j, and* $\mathrm{proj.dim}_A I^j = l-1$ *if and only if $j > a^2(G) - a(F)$.*

Finally, we show that the graded minimal free resolutions of the powers of an ideal also have a uniform behaviour. For equigenerated ideals, we can prove that the shifts which arise in the minimal resolutions are linear functions asymptotically and the Betti numbers are polynomial functions asymptotically. More explicitly,

Proposition 40 [Proposition 6.3.6] *Let I be a homogeneous ideal generated in degree d. Set $l = l(I)$, $s = n - \operatorname{depth}_{(\mathfrak{m}R)}(R)$. Then there is a finite set of integers $\{\alpha_{pi} \mid 0 \leq p \leq s, 1 \leq i \leq k_p\}$ and polynomials $\{Q_{\alpha_{pi}}(j) : 0 \leq p \leq s, 1 \leq i \leq k_p\}$ of degree $\leq l-1$ such that the graded minimal free resolution of I^j for j large enough is*

$$0 \to D_s^j \to \ldots \to D_0^j \to I^j \to 0\ ,$$

with $D_p^j = \bigoplus_i A(-\alpha_{pi} - dj)^{\beta_{pi}^j}$ and $\beta_{pi}^j = Q_{\alpha_{pi}}(j)$.

From this result, we may deduce that a finite number of the graded minimal free resolutions of the powers of an ideal determine the rest of them. This finite set of resolutions can be found for ideals with a very particular behaviour. For instance, we get

Proposition 41 [Proposition 6.3.10] *Let I be an equigenerated homogeneous ideal, and $b = a_*^2(R_A(I)) + l(I)$. If the graded minimal free resolutions of $I, I^2, \ldots, I^b$ are linear, then the graded minimal free resolutions of I^j are also linear for any j. Furthermore, the minimal free resolutions of $I, I^2, \ldots, I^b$ determine the minimal graded free resolutions of I^j for any j.*

Some parts of this work have already appeared published in:

- O. Lavila–Vidal, *On the Cohen-Macaulay property of diagonal subalgebras of the Rees algebra*, manuscripta math. 95 (1998), 47–58.

- O. Lavila–Vidal, S. Zarzuela, *On the Gorenstein property of the diagonals of the Rees algebra*, Collect. Math. 49, 2-3 (1998), 383–397.

It is a pleasure to thank my thesis advisor Santiago Zarzuela for his help, guidance and support throughout these years.

I would also thank the late Pr.Dr. Manfred Herrmann and his Seminar for their hospitality while I was visiting the Mathematisches Institut der Universität zu Köln.

I also wish to thank my fellows at the Departament d'Àlgebra i Geometria and at the Seminari d'Àlgebra Commutativa, especially Maria Alberich.

Finally, let me thank my friends, my family and Eduard for their love, patience and encouragement.

Chapter 1

Multigraded rings

In this chapter we collect some basic definitions and facts of the theory of multigraded rings which we will need in the next chapters. We also state the multigraded versions of some well-known results in the category of graded rings. Rings are always assumed to be noetherian.

1.1 Multigraded rings and modules

The general theory of multigraded rings and modules is analogous to that of graded rings and modules. We first recall some basic definitions. The main sources are [BH1], [HHR] and [GW1].

We use the following multi-index notation. For $\mathbf{n} = (n^1, \dots, n^r) \in \mathbb{Z}^r$, we set $|\mathbf{n}| = n^1 + \ldots + n^r$, and for $\mathbf{n}, \mathbf{m} \in \mathbb{Z}^r$, we define their sum $\mathbf{n} + \mathbf{m} = (n^1 + m^1, \dots, n^r + m^r)$, and we set $\mathbf{n} < \mathbf{m}$ ($\mathbf{n} \leq \mathbf{m}$) if $n^i < m^i$ ($n^i \leq m^i$) for every i.

A $\mathbb{Z}^r$-graded ring (or r-graded ring) is a ring S endowed with a direct sum decomposition $S = \bigoplus_{\mathbf{n} \in \mathbb{Z}^r} S_{\mathbf{n}}$, such that $S_{\mathbf{n}} S_{\mathbf{m}} \subset S_{\mathbf{n}+\mathbf{m}}$ for all $\mathbf{n}, \mathbf{m} \in \mathbb{Z}^r$. An r-graded S-module is an S-module M endowed with a decomposition $M = \bigoplus_{\mathbf{n} \in \mathbb{Z}^r} M_{\mathbf{n}}$, such that $S_{\mathbf{n}} M_{\mathbf{m}} \subset M_{\mathbf{n}+\mathbf{m}}$ for all $\mathbf{n}, \mathbf{m} \in \mathbb{Z}^r$. We shall call $M_{\mathbf{n}}$ the homogeneous component of M of degree $\mathbf{n}$. An element $x \in M$ is homogeneous of degree $\mathbf{n}$ if $x \in M_{\mathbf{n}}$. The degree of x is then denoted by $\deg x$. For any r-graded S-module M, we define the support of M to be the set $\operatorname{supp} M = \{\mathbf{n} \in \mathbb{Z}^r \mid M_{\mathbf{n}} \neq 0\}$.

For a given r-graded ring S, we may consider the category of r-graded S-modules $M^r(S)$. Its objects are the r-graded S-modules, and a morphism $f : M \to N$ in $M^r(S)$ is an S-module morphism such that $f(M_{\mathbf{n}}) \subset N_{\mathbf{n}}$ for all $\mathbf{n} \in \mathbb{Z}^r$.

Given an r-graded S-module M, an r-graded submodule is a submodule $N \subset M$ such that $N = \bigoplus_{\mathbf{n} \in \mathbb{Z}^r} N \cap M_{\mathbf{n}}$, equivalently, N is generated by homogeneous elements. The r-graded submodules of S are called homogeneous ideals. For an arbitrary ideal I of S, the homogeneous ideal I^* is defined to be the ideal generated by all the homogeneous elements of I.

As a first example of r-graded ring we have the polynomial ring $S = A[X_1, \ldots, X_n]$ defined over an arbitrary ring A. For every choice of elements $\mathbf{d}_1, \ldots, \mathbf{d}_n \in \mathbb{Z}^r$, we have a unique r-grading on S such that $\deg X_i = \mathbf{d}_i$ and $\deg a = 0$ for all $a \in A$.

For an r-graded S-module M and $\mathbf{k} \in \mathbb{Z}^r$, then $M(\mathbf{k})$ denotes the S-module M with the grading given by $M(\mathbf{k})_{\mathbf{n}} = M_{\mathbf{k}+\mathbf{n}}$.

If M, N are r-graded S-modules, we denote by $\underline{\mathrm{Hom}}_S(M, N)_0$ the abelian group of all the homomorphisms of r-graded S-modules from M into N. We set $\underline{\mathrm{Hom}}_S(M, N) = \bigoplus_{\mathbf{n} \in \mathbb{Z}^r} \underline{\mathrm{Hom}}_S(M, N(\mathbf{n}))_0$. Note that $\underline{\mathrm{Hom}}_S(M, N)_{\mathbf{k}}$ is nothing but the abelian group of S-module homomorphisms $f : M \to N$ such that $f(M_{\mathbf{n}}) \subset N_{\mathbf{n}+\mathbf{k}}$ for all $\mathbf{n} \in \mathbb{Z}^r$. The derived functors of $\underline{\mathrm{Hom}}_S(\ ,\ )$ are $\underline{\mathrm{Ext}}^i_S(\ ,\ )$, with $i \in \mathbb{N}$.

1.2 Multigraded cohomology

Next we are going to introduce the local cohomology functor in the category of multigraded modules, mainly following [HHR]. The basic results are the multigraded version of the Local Duality Theorem and the good behaviour of the local cohomology modules under a change of grading.

From now on in this chapter, we assume that $S = \bigoplus_{\mathbf{n} \in \mathbb{N}^r} S_{\mathbf{n}}$ is an r-graded ring defined over a local ring $S_0 = A$. Then S has a unique homogeneous maximal ideal $\mathcal{M} = \mathfrak{m} \oplus (\bigoplus_{\mathbf{n} \neq 0} S_{\mathbf{n}})$, where $\mathfrak{m}$ is the maximal ideal of A. Set $d = \dim S$.

If $I \subset S$ is a homogeneous ideal and M is an r-graded S-module, we denote by $\underline{\mathrm{H}}^0_I(M) = \Gamma_I(M) = \{x \in M : I^k x = 0 \text{ for some } k \geq 0\}$. Note that $\underline{\mathrm{H}}^0_I(M)$

is an r-graded submodule of M. The local cohomology functors $\underline{\mathrm{H}}^i_I(\ )$ are the right derived functors of $\Gamma_I(\ )$ in the category of r-graded S-modules. If no confusion, we will usually denote them by $H^i_I(\ )$.

An r-graded S-module K_S is called a canonical module of S if

$$K_S \otimes_A \widehat{A} \cong \underline{\mathrm{Hom}}_S(\underline{\mathrm{H}}^d_{\mathcal{M}}(S), \underline{\mathrm{E}}_S(k))\,,$$

where k is the residue field of A and $\underline{\mathrm{E}}_S(k)$ is the injective envelope of k in the category of r-graded S-modules. The injective envelope $\underline{\mathrm{E}}_S(k)$ of k is $\underline{\mathrm{Hom}}_A(S, \mathrm{E}_A(k))$, where A is thought as an r-graded ring concentrated in degree 0, and both S and $\mathrm{E}_A(k)$ are considered as r-graded A-modules. Therefore, we have

$$K_S \otimes_A \widehat{A} \cong \underline{\mathrm{Hom}}_A(\underline{\mathrm{H}}^d_{\mathcal{M}}(S), \mathrm{E}_A(k)) = \bigoplus_{\mathbf{n}\in\mathbb{Z}^r} \mathrm{Hom}_A([\underline{\mathrm{H}}^d_{\mathcal{M}}(S)]_{-\mathbf{n}}, E_A(k)).$$

If a canonical module exists, it is finitely generated and unique up to an isomorphism. In the particular case where $A = k$ is a field, the canonical module of S exists and

$$K_S \cong \underline{\mathrm{Hom}}_k(\underline{\mathrm{H}}^d_{\mathcal{M}}(S), k).$$

The next results are the extension to the r-graded case of two of the main properties of the canonical module, well-known for the graded case (see [GW2, Theorem 2.2.2]).

Theorem 1.2.1 *(Local Duality) Let S be an r-graded ring defined over a complete local ring A. Let $\mathcal{M}$ be the homogeneous maximal ideal of S. Then S is Cohen-Macaulay if and only if every finitely generated r-graded S-module M satisfies*

$$\underline{\mathrm{Hom}}_S(\underline{\mathrm{H}}^i_{\mathcal{M}}(M), \underline{\mathrm{E}}_S(k)) \cong \underline{\mathrm{Ext}}^{d-i}_S(M, K_S)\ ,\ i = 0, \ldots, d.$$

Corollary 1.2.2 *Let S be a Cohen-Macaulay r-graded ring with canonical module K_S. Let T be an r-graded ring defined over a local ring such that there exists a finite r-graded ring morphism $S \to T$. Then T has canonical module*

$$K_T = \underline{\mathrm{Ext}}^e_S(T, K_S)\,,$$

where $e = \dim S - \dim T$.

Often we are going to consider the ring S endowed with a different grading obtained in the following way: given a group morphism $\varphi : \mathbb{Z}^r \to \mathbb{Z}^q$ such that $\varphi(\text{supp } S) \subset \mathbb{N}^q$, we can define the $\mathbb{N}^q$-graded ring

$$S^\varphi := \bigoplus_{\mathbf{m}\in\mathbb{N}^q} \Big(\bigoplus_{\varphi(\mathbf{n})=\mathbf{m}} S_{\mathbf{n}} \Big).$$

Similarly, given an r-graded S-module M, we may define the q-graded S^φ-module M^φ as

$$M^\varphi := \bigoplus_{\mathbf{m}\in\mathbb{Z}^q} \Big(\bigoplus_{\varphi(\mathbf{n})=\mathbf{m}} M_{\mathbf{n}} \Big).$$

Then $(\ )^\varphi : M^r(S) \to M^q(S^\varphi)$ is an exact functor. By considering $\varphi_j : \mathbb{Z}^r \to \mathbb{Z}$ the projection on the j-component, that is, $\varphi_j(\mathbf{n}) = n^j$, we denote by $S_j = S^{\varphi_j}$ and by $M_j = M^{\varphi_j}$. Note that S_j is just the ring S graded by the j-th partial degree.

The next lemma shows that the local cohomology modules behave well under a change of grading.

Lemma 1.2.3 *[HHR, Lemma 1.1] Let S be an r-graded ring defined over a local ring. Let $\mathcal{M}$ be the homogeneous maximal ideal of S. Let $\varphi : \mathbb{Z}^r \to \mathbb{Z}^q$ be a morphism such that $\varphi(\operatorname{supp} S) \subset \mathbb{N}^q$. For every r-graded S-module L, we have*

$$\underline{H}^i_{\mathcal{M}}(L)^\varphi = \underline{H}^i_{\mathcal{M}^\varphi}(L^\varphi),\ \forall i.$$

1.3 Multigraded a-invariants

We begin this section by extending the definition of the a-invariants of a graded module to the multigraded case. After that, under some mild assumptions, we relate the multigraded a-invariants of a multigraded module to the shifts which appear in its multigraded minimal free resolution. This result will be essential in the next chapters. In the graded case, a similar result can be found in [BH1, Example 3.6.15] for Cohen-Macaulay modules.

Let S be a d-dimensional $\mathbb{N}^r$-graded ring defined over a local ring. For each $i = 0, ..., d$, the multigraded a_i-invariant of S is $\mathbf{a}_i(S) = (a_i^1(S), ..., a_i^r(S))$, where

$$a_i^j(S) = \max\{m \in \mathbb{Z} \mid \exists \mathbf{n} \in \mathbb{Z}^r : \varphi_j(\mathbf{n}) = m, [\underline{\mathrm{H}}^i_{\mathcal{M}}(S)]_{\mathbf{n}} \neq 0\}$$

if $\underline{\mathrm{H}}^i_{\mathcal{M}}(S) \neq 0$ and $a^j_i(S) = -\infty$ otherwise. We will denote by $\mathbf{a}(S) = \mathbf{a}_d(S)$. Note that by Lemma 1.2.3

$$a^j_i(S) = \max\{m \in \mathbb{Z} \mid [\underline{\mathrm{H}}^i_{\mathcal{M}_j}(S_j)]_m \neq 0\} = a_i(S_j).$$

Following N.V. Trung [Tr2], the multigraded a_*-invariant of S is defined as $\mathbf{a}_*(S) = (a^1_*(S), ..., a^r_*(S))$, where $a^j_*(S) = \max\{a^j_0(S), \ldots, a^j_d(S)\}$. Similarly, for any finitely generated r-graded S-module M we may define the a-invariants $\mathbf{a}_i(M)$ of M and the a_*-invariant $\mathbf{a}_*(M)$ of M.

Observe that if there exists K_S the canonical module of S , then

$$a^j(S) = a^j_d(S) = -\min\{m \in \mathbb{Z} \mid \exists \mathbf{n} \in \mathbb{Z}^r : \varphi_j(\mathbf{n}) = m, [K_S]_{\mathbf{n}} \neq 0\}.$$

If S has a canonical module K_S, S is said to be quasi-Gorenstein if there exists an r-graded isomorphism $K_S \cong S(\mathbf{a}(S))$, and Gorenstein if in addition S is Cohen-Macaulay.

From now on in this section we assume that S is a noetherian $\mathbb{N}^r$-graded algebra defined over a field k, and let $\mathcal{M}$ be its homogeneous maximal ideal. Our main purpose is then to compute the multigraded a-invariants of a finitely generated r-graded S-module M from an r-graded minimal finite free resolution of M over S, whenever it exists and S is Cohen-Macaulay. To begin with, let us consider

$$\ldots \to D_t \to \ldots \to D_1 \to D_0 \to 0$$

an exact sequence of finitely generated r-graded S-modules such that $\mathrm{Im}(D_{p+1}) \subset \mathcal{M}D_p$, for all $p \geq 0$. Let us denote by $\{\mathbf{v}_{pq}\}$ the set of degree vectors of a minimal homogeneous system of generators of D_p. Note that this set is uniquely determined because it can be obtained as the homogeneous components of the vector space $D_p \otimes_S k$ which are not zero. We set $\mathbf{m}_p = \min_{\leq_{lex}}\{\mathbf{v}_{pq}\}$ and $\mathbf{M}_p = \max_{\leq_{lex}}\{\mathbf{v}_{pq}\}$, where $\leq_{lex}$ is the lexicographic order. Let us denote by $n^j_p = \min_q\{v^j_{pq}\}$, $t^j_p = \max_q\{v^j_{pq}\}$, where $\mathbf{v}_{pq} = (v^1_{pq}, \ldots, v^r_{pq})$, and $\mathbf{n}_p = (n^1_p, ..., n^r_p)$, $\mathbf{t}_p = (t^1_p, ..., t^r_p)$. Let us also consider $\leq$ the partial order in $\mathbb{Z}^r$ defined coefficientwise. Then we have

Lemma 1.3.1 *(i)* $\mathbf{n}_p \leq \mathbf{n}_{p+1}$.

(ii) $\mathbf{m}_p <_{lex} \mathbf{m}_{p+1}$

Proof. Let $C_p = \mathrm{Coker}(D_{p+1} \to D_p)$, $\forall p \geq 1$. Then there are short exact sequences

$$0 \to C_{p+2} \to D_{p+1} \to C_{p+1} \to 0, \ \ \forall p \geq 0.$$

Applying the functor $-\bigotimes_S k$, we get exact sequences

$$C_{p+2}/\mathcal{M}C_{p+2} \to D_{p+1}/\mathcal{M}D_{p+1} \to C_{p+1}/\mathcal{M}C_{p+1} \to 0, \ \ \forall p \geq 0.$$

Since $C_{p+2} \subset \mathcal{M}D_{p+1}$, then the first map is the zero morphism. Therefore we get isomorphisms

$$D_{p+1}/\mathcal{M}D_{p+1} \xrightarrow{\cong} C_{p+1}/\mathcal{M}C_{p+1}.$$

Let us denote by $\{e_{pq}\}$ a minimal homogeneous system of generators of D_p with $\deg(e_{pq}) = \mathbf{v}_{pq}$, and let f be the map from D_{p+1} to D_p. From the isomorphism it follows that $f(e_{p+1,q}) \neq 0$, for all q. Now let us fix q. We can write $f(e_{p+1,q}) = \sum_l \lambda_l e_{pl}$, where λ_l are homogeneous elements of $\mathcal{M}$. Set $\deg(\lambda_l) = (\lambda_l^1, ..., \lambda_l^r) \in \mathbb{N}^r$ and note that $\deg(\lambda_l) \neq 0$ if $\lambda_l \neq 0$. Looking at the j-th component of the degree, we get $v^j_{p+1,q} \geq \min_l\{ v^j_{pl}\} = n^j_p$, and so $n^j_{p+1} \geq n^j_p$ for all j.

To obtain (ii), it is enough to prove that $\mathbf{v}_{p+1,q} >_{lex} \mathbf{m}_p$ for all q. We have already shown that $v^1_{p+1,q} \geq \min_l\{ v^1_{pl}\} = m^1_p$. If $v^1_{p+1,q} > m^1_p$, we are done. Otherwise, $v^1_{p+1,q} = m^1_p$ and so $\lambda^1_l = 0$ for each l such that $\lambda_l \neq 0$. Then we have $v^2_{p+1,q} \geq \min_l\{ v^2_{pl} \mid v^1_{pl} = m^1_p\} = m^2_p$. By repeating this argument, we get the result since there exist l, j such that $\lambda^j_l > 0$. $\square$

Let S be a d-dimensional r-graded Cohen-Macaulay k-algebra. Assume that M is a finitely generated r-graded S-module with a finite minimal r-graded free resolution over S

$$0 \to D_l \to \ldots \to D_1 \to D_0 \to M \to 0,$$

with $D_p = \bigoplus_q S(a^1_{pq}, ..., a^r_{pq})$. Set $m = \dim M$, $\rho = \operatorname{depth} M$. Note that $l = d - \rho$ by the graded Auslander-Buchsbaum formula. Next we are going to study the shifts which appear in this resolution.

Note that, with the notation introduced before,

$$\begin{aligned} n^j_p &= \min_q\{-a^j_{pq}\}, \\ t^j_p &= \max_q\{-a^j_{pq}\}, \\ \mathbf{m}_p &= \min_{\leq_{lex}}\{(-a^1_{pq}, ..., -a^r_{pq})\}, \end{aligned}$$

$$\mathbf{M}_p = \max_{\leq_{lex}}\{\,(-a^1_{pq},...,-a^r_{pq})\}.$$

We will also denote by $t^j_p(M) = t^j_p$, $t^j_*(M) = \max\{t^j_0,\dots,t^j_l\}$, $\mathbf{t}_*(M) = (t^1_*(M),\dots,t^r_*(M))$. From Lemma 1.3.1, we have $\mathbf{n}_p \leq \mathbf{n}_{p+1}$, $\mathbf{m}_p <_{lex} \mathbf{m}_{p+1}$. Furthermore,

Lemma 1.3.2 *(i)* $\mathbf{M}_0 <_{lex} \mathbf{M}_1 <_{lex} \cdots <_{lex} \mathbf{M}_{d-m-1} <_{lex} \mathbf{M}_{d-m}$.

(ii) $\mathbf{t}_0 \leq \mathbf{t}_1 \leq \cdots \leq \mathbf{t}_{d-m-1} \leq \mathbf{t}_{d-m}$.

Proof. Let K_S be the canonical module of S. Note that it exists because S is a finitely generated k-algebra. By setting $C_p = \operatorname{Coker}(D_{p+1} \to D_p)$ for $p \geq 0$, we get short exact sequences

$$0 \to C_{p+1} \to D_p \to C_p \to 0,$$

for $0 \leq p \leq l-1$, where $C_0 = M$, $C_l = D_l$. For any $p < d-m-1$, we have

$$\underline{\operatorname{Ext}}^1_S(C_p, K_S) \cong \underline{\operatorname{Ext}}^2_S(C_{p-1}, K_S) \cong \cdots \cong \underline{\operatorname{Ext}}^{p+1}_S(M, K_S) = 0$$

by Theorem 1.2.1. Therefore, by applying the functor $(\ )^* = \underline{\operatorname{Hom}}_S(\ , K_S)$ to the sequences above for $p \leq d-m-1$, we get exact sequences

$$0 \to C^*_p \to D^*_p \to C^*_{p+1} \to 0, \ \text{ for } p \leq d-m-2,$$

$$0 \to C^*_{d-m-1} \to D^*_{d-m-1} \to C^*_{d-m} \to H^m_{\mathcal{M}}(M)^\vee \to 0,$$

where $(\ )^\vee = \underline{\operatorname{Hom}}_k(\ , k)$. By gluing these exact sequences, we get the r-graded exact sequence

$$0 \to D^*_0 \to \ldots \to D^*_{d-m-1} \to C^*_{d-m} \to H^m_{\mathcal{M}}(M)^\vee \to 0.$$

Observe that $D^*_p = \bigoplus_q K_S(-a^1_{pq},...,-a^r_{pq})$. One can also check that $\operatorname{Im}(D^*_p) \subset \mathcal{M}D^*_{p+1}$ for all $p \leq d-m-2$.

Let $\{\,\mathbf{b}_1,...,\mathbf{b}_k\}$ be the set of degree vectors of a minimal homogeneous system of generators of K_S. If we denote by $\mathbf{a}_{pq} = (a^1_{pq},...,a^r_{pq})$, then the vectors $\mathbf{a}_{pq}+\mathbf{b}_i$ are the degrees of a minimal homogeneous system of generators of D^*_p. For $p \leq d-m-1$, let us consider

$$\widetilde{\mathbf{m}}_p = \min_{\leq_{lex}}\{\,\mathbf{a}_{pq}+\mathbf{b}_i\} = -\mathbf{M}_p + \min_{\leq_{lex}}\{\,\mathbf{b}_i\}\ ,$$

$$\tilde{n}^j_p = \min_{q,i}\{\,a^j_{pq}+b^j_i\} = -t^j_p + \min_i\{\,b^j_i\}\ .$$

According to Lemma 1.3.1, we have $\widetilde{n}^j_{p+1} \leq \widetilde{n}^j_p$ and $\widetilde{\mathbf{m}}_{p+1} <_{lex} \widetilde{\mathbf{m}}_p$, so

$$\mathbf{t}_0 \leq \mathbf{t}_1 \leq \cdots \leq \mathbf{t}_{d-m-2} \leq \mathbf{t}_{d-m-1}$$
$$\mathbf{M}_0 <_{lex} \mathbf{M}_1 <_{lex} \cdots <_{lex} \mathbf{M}_{d-m-2} <_{lex} \mathbf{M}_{d-m-1}.$$

Next we want to show that $\mathbf{M}_{d-m} >_{lex} \mathbf{M}_{d-m-1}$. To this end, let us study the morphism $D_{d-m} \to D_{d-m-1}$ and for that denote by $\nu : C_{d-m} \to D_{d-m-1}$. Assume that there is an element u in the basis of D_{d-m-1} of degree $\mathbf{M}_{d-m-1} \geq_{lex} \mathbf{M}_{d-m}$. If g is a homogeneous minimal generator of C_{d-m}, then g has trivial terms in u: Otherwise, we would have that $\mathbf{M}_{d-m-1} <_{lex} \deg g \leq_{lex} \mathbf{M}_{d-m}$ because $C_{d-m} \subset \mathcal{M}D_{d-m-1}$. Let $\mathbf{b} = \min_{<_{lex}}\{\mathbf{b}_1, ..., \mathbf{b}_k\}$, and let us take $c \in [K_S]_{\mathbf{b}}$, $c \neq 0$. Let $w : D_{d-m-1} \to K_S$ defined by $w(u) = c$, $w(v) = 0$ for any $v \neq u$ homogeneous element in the basis of D_{d-m-1}. Then $\nu^* : D^*_{d-m-1} \to C^*_{d-m}$ satisfies $\nu^*(w) = 0$, hence ν^* is not a monomorphism in degree $\deg w = \deg w(u) - \deg(u) = \mathbf{b} - \mathbf{M}_{d-m-1}$. Therefore $[C^*_{d-m-1}]_{\mathbf{b}-\mathbf{M}_{d-m-1}} \neq 0$, and then $[D^*_{d-m-2}]_{\mathbf{b}-\mathbf{M}_{d-m-1}} \neq 0$, so there exists a shift $\mathbf{a} = (a^1, \ldots, a^r)$ in D_{d-m-2} such that $-\mathbf{a} \geq_{lex} \mathbf{M}_{d-m-1}$. So we obtain $\mathbf{M}_{d-m-2} \geq_{lex} \mathbf{M}_{d-m-1}$ which is a contradiction.

Furthermore, note that the first component of $\mathbf{M}_p$ is t^1_p. Therefore, we have $t^1_{d-m-1} \leq t^1_{d-m}$ since $\mathbf{M}_{d-m-1} <_{lex} \mathbf{M}_{d-m}$. The inequalities $t^j_{d-m-1} \leq t^j_{d-m}$ for $j = 2, \ldots, r$ follow directly from the next remark. $\square$

Remark 1.3.3 Given a permutation σ of $\{1, \ldots, r\}$, we may define $\leq_\sigma$ to be the order in $\mathbb{Z}^r$ defined by

$$(u_1, ..., u_r) \leq_\sigma (v_1, ..., v_r) \iff (u_{\sigma(1)}, ..., u_{\sigma(r)}) \leq_{lex} (v_{\sigma(1)}, ..., v_{\sigma(r)}).$$

Then Lemmas 1.3.1 and 1.3.2 also hold if we define

$$\mathbf{m}^\sigma_p = \min_{\leq_\sigma}\{(-a^1_{pq}, ..., -a^r_{pq})\},$$
$$\mathbf{M}^\sigma_p = \max_{\leq_\sigma}\{(-a^1_{pq}, ..., -a^r_{pq})\}.$$

The following result gives a formula for the multigraded a_*-invariant of M by means of the shifts which arise in its resolution over S (see [BH1, Example 3.6.15] for the case of a $\mathbb{Z}$-graded Cohen-Macaulay module).

Theorem 1.3.4 *For each $j = 1, \ldots, r$, we have*

(i) $a^j_{d-p}(M) \leq t^j_p(M) + a^j(S)$, *for* $d - m \leq p \leq d - \rho$.

(ii) Assume that for some p there exists σ s.t. $\sigma(1) = j$ and $\mathbf{M}_p^\sigma >_\sigma \mathbf{M}_{p+1}^\sigma$. Then $a_{d-p}^j(M) = t_p^j(M) + a^j(S)$.

(iii) $a_*^j(M) = t_*^j(M) + a^j(S)$. That is, $\mathbf{a}_*(M) = \mathbf{t}_*(M) + \mathbf{a}(S)$.

Proof. From the minimal r-graded free resolution of M over S

$$0 \to D_l \to \ldots \to D_0 \to M \to 0,$$

by setting $C_p = \operatorname{Coker}(D_{p+1} \to D_p)$, we have short exact sequences

$$0 \to C_{p+1} \to D_p \to C_p \to 0,$$

for $0 \leq p \leq l-1$. By Theorem 1.2.1, if we apply the functor $(\ )^* = \underline{\operatorname{Hom}}_S(\ , K_S)$ to the sequences above we get exact sequences

$$(1)\ \ 0 \to D_0^* \to \ldots \to D_{d-m-1}^* \to C_{d-m}^* \to H_{\mathcal{M}}^m(M)^\vee \to 0,$$

and

$$(2)\ \ 0 \to C_p^* \to D_p^* \to C_{p+1}^* \to 0, \ \ \text{for } p \leq d-m-2,$$

$$(3)\ \ 0 \to C_{p-1}^* \to D_{p-1}^* \to C_p^* \to H_{\mathcal{M}}^{d-p}(M)^\vee \to 0, \ \ \text{for } p \geq d-m\,,$$

where $(\ )^\vee = \underline{\operatorname{Hom}}_k(\ , k)$. Note that for $d-m \leq p \leq d-\rho$ we have monomorphisms

$$0 \to C_p^* \to D_p^* = \bigoplus_q K_S(-a_{pq}^1, \ldots, -a_{pq}^r),$$

and so $[C_p^*]_{-\mathbf{i}} = 0$ for any $\mathbf{i}$ such that $i^1 > t_p^1 + a^1(S)$. Now from the epimorphisms

$$C_p^* \to H_{\mathcal{M}}^{d-p}(M)^\vee \to 0,$$

we get $H_{\mathcal{M}}^{d-p}(M)_{\mathbf{i}} = 0$ if $i^1 > t_p^1 + a^1(S)$, and therefore $a_{d-p}^1(M) \leq t_p^1(M) + a^1(S)$. This proves (i) for the case $j = 1$.

Assume now that there exists p with $\mathbf{M}_p >_{lex} \mathbf{M}_{p+1}$ (then $p \geq d-m$ according to Lemma 1.3.2). Let $\mathbf{b} = (b^1, \ldots, b^r)$ be the minimum with respect to the lexicographic order such that $[K_S]_{\mathbf{b}} \neq 0$. Note that $b^1 = -a^1(S)$. Let $\mathbf{i} = \mathbf{M}_p - \mathbf{b}$. Since $[D_{p+1}^*]_{-\mathbf{i}} = 0$ because $\mathbf{M}_p >_{lex} \mathbf{M}_{p+1}$, we have $[C_{p+1}^*]_{-\mathbf{i}} = 0$ by (3), and so $[C_p^*]_{-\mathbf{i}} = [D_p^*]_{-\mathbf{i}}$ also by (3). Then, denoting by $f : D_p \to D_{p-1}$, we get an exact sequence

$$[D_{p-1}^*]_{-\mathbf{i}} \xrightarrow{f^*} [D_p^*]_{-\mathbf{i}} \to [H_{\mathcal{M}}^{d-p}(M)]_{\mathbf{i}} \to 0.$$

Let $e_1, \ldots, e_s$ be the elements of the canonical basis of D_p with degree $\mathbf{M}_p$, and let $v_1, \ldots, v_m$ be the canonical basis of D_{p-1}. Since $f^*(D^*_{p-1}) \subset \mathcal{M}D^*_p$ and $[D^*_p]_{-\mathbf{i}} = [K_S]_{\mathbf{b}} e^*_1 \oplus \ldots \oplus [K_S]_{\mathbf{b}} e^*_s$, we have that $[K_S]_{\mathbf{b}} e^*_1 \not\subset \mathrm{Im} f^*$. In particular, f^* is not an epimorphism, and so $[H^{d-p}_{\mathcal{M}}(M)]_{\mathbf{i}} \neq 0$. Therefore, $a^1_{d-p}(M) \geq i^1 = M^1_p - b^1 = t^1_p(M) + a^1(S)$. This proves (ii) for the case $j = 1$, $\sigma = Id$.

Let p be the greatest integer such that $\mathbf{M}_p = \max_{\leq_{lex}} \{\mathbf{M}_0, \ldots, \mathbf{M}_l\}$. Then, $\mathbf{M}_{p+1} <_{lex} \mathbf{M}_p$, so $a^1_{d-p}(M) = t^1_p(M) + a^1(S)$ by (ii). Therefore, $a^1_*(M) = t^1_*(M) + a^1(S)$ and we have (iii) for $j = 1$. The proof of the statement for $j = 2, ..., r$ follows from Remark 1.3.3. $\square$

1.4 Scheme associated to a multigraded ring

Let S be a noetherian $\mathbb{N}^r$-graded ring. We call S standard if S may be generated over S_0 by elements in degrees $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$. Similarly to the graded case, we may associate to such a ring a multigraded scheme in a natural way (see [Hy]). Our purpose is to extend this construction to a more general class of multigraded rings, which will recover the standard case as well as the Rees algebra of any homogeneous ideal in a graded k-algebra.

Let S be a noetherian $\mathbb{N}^r$-graded ring finitely generated over S_0 by homogeneous elements $x_{11}, \ldots, x_{1k_1}, \ldots, x_{r1}, \ldots, x_{rk_r}$ of degrees $\deg(x_{ij}) = (d^1_{ij}, \ldots, d^{i-1}_{ij}, 1, 0, \ldots, 0)$, where d^l_{ij} are non-negative integers, and set $d^l_i = \max_j \{d^l_{ij}\}$. This class of rings includes for instance any standard $\mathbb{N}^r$-graded ring by taking $d^l_{ij} = 0$. For every $i = 1, \ldots, r$, let I_i be the ideal of S generated by the homogeneous components of S of degree $\mathbf{n} = (n_1, \ldots, n_r)$ such that $n_i > 0, n_{i+1} = \ldots = n_r = 0$. Then we define the irrelevant ideal of S as $S_+ = I_1 \cdots I_r$. We are going to associate a scheme to S in the following way. A homogeneous prime ideal P of S is said to be relevant if P does not contain S_+. Then we define the set $\mathrm{Proj}^{\,r}(S)$ to be the set of all relevant homogeneous prime ideals P. It is easy to check that $\dim S/P \geq r$ for any relevant prime ideal (see the proof of Lemma 1.4.1). Following [STV] (where the standard bigraded case was studied), we define the relevant dimension of S as

$$\mathrm{rel.dim}\, S = \begin{cases} r - 1 & \text{if } \mathrm{Proj}^{\,r}(S) = \emptyset \\ \max\{\dim S/P \mid P \in \mathrm{Proj}^{\,r}(S)\} & \text{if } \mathrm{Proj}^{\,r}(S) \neq \emptyset \end{cases}.$$

If I is a homogeneous ideal of S, we define the subset $V_+(I) := \{P \in \mathrm{Proj}\,^r(S) \mid I \subset P\}$. We can define a topology on $\mathrm{Proj}\,^r(S)$ by taking as closed subsets the subsets of the form $V_+(I)$. Next, to define a sheaf of rings $\mathcal{O}$ in $\mathrm{Proj}\,^r(S)$, we first consider for each $P \in \mathrm{Proj}\,^r(S)$ the homogeneous localization by P

$$S_{(P)} = \{\, \frac{a}{s} \mid s \notin P,\ a, s \in S_{\mathbf{n}},\ \mathbf{n} \in \mathbb{Z}^r \,\}.$$

For any open subset $U \subset \mathrm{Proj}\,^r(S)$, we define $\mathcal{O}(U)$ to be the set of functions $t : U \to \bigsqcup_{P \in U} S_{(P)}$ such that for each $P \in U$, $t(P) \in S_{(P)}$ and t is locally a quotient of elements of S. Then, $\mathcal{O}$ is a sheaf of rings. We call $\mathrm{Proj}\,^r(S)$ the r-projective scheme associated to S. Defining for any homogeneous $f \in S_+$ the set $D_+(f) = \{P \in \mathrm{Proj}\,^r(S) \mid f \notin P\}$ we have an open cover of $\mathrm{Proj}\,^r(S)$, and for each such open set we have an isomorphism of locally ringed spaces

$$(D_+(f), \mathcal{O}|D_+(f)) \cong \mathrm{Spec}(S_{(f)}).$$

Moreover, $\mathcal{O}_P \cong S_{(P)}$ for any relevant prime ideal P, hence $\mathrm{Proj}\,^r(S)$ is a scheme in a natural way. This construction extends the usual one given in the standard case (see [Hy]).

The next lemma computes the dimension of $\mathrm{Proj}\,^r(S)$. Its proof follows the same arguments as in [Hy, Lemma 1.2], but we include it for completeness.

Lemma 1.4.1 $\dim \mathrm{Proj}\,^r(S) = \mathrm{rel.dim}\, S - r$.

Proof. We may assume that $\mathrm{Proj}\,^r(S) \neq \emptyset$ (otherwise the result is trivial). Let $P \in \mathrm{Proj}\,^r(S)$ be a closed point. Since the projection $\mathrm{Proj}\,^r(S) \to \mathrm{Spec}(S_0)$ is proper, we have that $P_0 = P \cap S_0$ is a closed point of $\mathrm{Spec}(S_0)$, so $(S/P)_0 = S_0/P_0$ is a field. Let us denote by $T = S/P$, and note that $\dim \mathrm{Proj}\,^r(T) = 0$. For $j = 1, \ldots, r$, let J_j be the ideal of T generated by the homogeneous components of T of degree $\mathbf{n}$ such that $n_j > 0, n_{j+1} = \ldots = n_r = 0$. We have a maximal chain of homogeneous prime ideals

$$0 \subset J_r \subset J_{r-1} + J_r \subset \ldots \subset J_1 + \ldots + J_r,$$

so $\dim T = r$ because T is a catenary ring. On the other hand, for a given minimal prime $Q_0 \in \mathrm{Proj}\,^r(S)$, we have a chain of homogeneous prime ideals of type $Q_0 \subset \ldots \subset Q_s \subset \ldots \subset Q_{s+r}$, with Q_s a closed point of $\mathrm{Proj}\,^r(S)$. Therefore,

$$\begin{aligned}\dim \operatorname{Proj}{}^r(S) &= \sup\{\operatorname{ht} Q : Q \in \operatorname{Proj}{}^r(S)\} \\ &= \sup\{\dim S/Q : Q \in \operatorname{Proj}{}^r(S)\} - r \\ &= \operatorname{rel.dim} S - r. \square\end{aligned}$$

Next we are going to define the diagonal functor. Given e_1,..., e_r positive integers, the set

$$\Delta := \{(e_1 s, ..., e_r s) \mid s \in \mathbb{Z}\}$$

is called the $(e_1, ..., e_r)$-diagonal of $\mathbb{Z}^r$. We may then define the diagonal of S along Δ as the graded ring $S_\Delta := \bigoplus_{s\in\mathbb{Z}} S_{(e_1 s,...,e_r s)}$. Similarly, given an r-graded S-module M we define the diagonal of M along Δ as the graded S_Δ-module $M_\Delta := \bigoplus_{s\in\mathbb{Z}} M_{(e_1 s,...,e_r s)}$. Then we have an exact functor

$$(\ )_\Delta : M^r(S) \to M^1(S_\Delta),$$

called diagonal functor.

Let us denote by $X = \operatorname{Proj}{}^r(S)$, and for each Δ, let $X_\Delta = \operatorname{Proj}(S_\Delta)$. By considering diagonals $\Delta = (e_1, \ldots, e_r)$ such that $e_r > 0$, $e_{r-1} > d_r^{r-1} e_r$, ..., $e_1 > d_2^1 e_2 + \ldots + d_r^1 e_r$, then the sheaf of ideals $\mathcal{L} = (S_{(e_1,...,e_r)})\, \mathcal{O}_X$ defines an isomorphism $X \xrightarrow{\cong} X_\Delta$. In particular, this isomorphism allows us to compute the dimension of S_Δ, extending [STV, Proposition 2.3] where this dimension was computed for bigraded standard k-algebras.

Lemma 1.4.2 *Assume that S_0 is an artinian local ring. Then* $\dim S_\Delta = \operatorname{rel.dim} S - r + 1$, *for any* $\Delta = (e_1, \ldots, e_r)$ *with* $e_r > 0$, $e_{r-1} > d_r^{r-1} e_r, \ldots, e_1 > d_2^1 e_2 + \ldots + d_r^1 e_r$.

Proof. From the isomorphism $X \cong X_\Delta$, we have that $\operatorname{rel.dim} S_\Delta = \operatorname{rel.dim} S - r + 1$ by Lemma 1.4.1. Moreover, since S_0 is artinian, any minimal prime ideal of S_Δ is relevant, and so $\operatorname{rel.dim} S_\Delta = \dim S_\Delta$. $\square$

Classically, S is the multihomogeneous coordinate ring of a multiprojective variety V contained in some multiprojective space $\mathbb{P}_k^{n_1} \times \ldots \times \mathbb{P}_k^{n_r}$. By taking the $(1, \ldots, 1)$-diagonal, S_Δ is then the homogeneous coordinate ring of the image of V via the Segre embedding $\mathbb{P}_k^{n_1} \times \ldots \times \mathbb{P}_k^{n_r} \to \mathbb{P}_k^N$, where $N = (n_1 + 1) \ldots (n_r + 1) - 1$.

1.5 Hilbert polynomial of multigraded modules

Let $S = \bigoplus_{\mathbf{n}\in\mathbb{N}^r} S_{\mathbf{n}}$ be an r-graded ring defined over an artinian local ring $S_0 = A$. If S is standard, then we have that the Hilbert function of any finitely generated r-graded S-module L, $H(L,\mathbf{n}) = \mathrm{length_A(L_{\mathbf{n}})}$, is a polynomial function; that is, there exists a polynomial $P_L(t_1,\dots,t_r) \in \mathbb{Q}[t_1,\dots,t_r]$, called Hilbert polynomial of L, such that for any $\mathbf{n} \gg 0$, $P_L(n_1,\dots,n_r) = \mathrm{length}_A(L_{\mathbf{n}})$ (see [HHRT], [KMV]). In this section we are going to extend the existence of such a polynomial for the larger class of finitely generated r-graded modules defined over the multigraded rings introduced in Section 1.4. Furthermore, we will state a formula for the difference between the Hilbert polynomial and the Hilbert function of any finitely generated r-graded module analogous to the one known in the graded case.

Let S be a noetherian $\mathbb{N}^r$-graded ring generated over $S_0 = A$ by homogeneous elements $x_{11},\dots,x_{1k_1},\dots,x_{r1},\dots,x_{rk_r}$ in degrees $\deg(x_{ij}) = (d_{ij}^1,\dots,d_{ij}^{i-1},1,0,\dots,0)$, where $d_{ij}^l \geq 0$. Set $d_i^l = \max_j\{d_{ij}^l\}$.

Given a finitely generated r-graded S-module L, let us define its homogeneous support as $\mathrm{Supp}_+(L) = \{\, P \in \mathrm{Proj}\,^r(S) \mid L_P \neq 0\}$. Note that $\mathrm{Supp}_+(L) = V_+(\mathrm{Ann}\, L)$ is a closed subset of $\mathrm{Proj}\,^r(S)$. We define the relevant dimension of L as

$$\mathrm{rel.dim}\, L = \begin{cases} r-1 & \text{if } \mathrm{Supp}_+(L) = \emptyset \\ \max\{\dim S/P \mid P \in \mathrm{Supp}_+(L)\} & \text{if } \mathrm{Supp}_+(L) \neq \emptyset \end{cases}.$$

One can check that $\mathrm{rel.dim}\, L = \dim \mathrm{Supp}_+ L + r$.

From now on in this section we will assume that A is an artinian local ring. Given a finitely generated r-graded S-module L, its homogeneous components $L_{\mathbf{n}}$ are finitely generated A-modules, and hence have finite length. The numerical function $H(L,\,\cdot\,) : \mathbb{Z}^r \to \mathbb{Z}$ with $H(L,\mathbf{n}) = \mathrm{length_A(L_{\mathbf{n}})}$ is the Hilbert function of L. Next result shows the existence of the Hilbert polynomial for any finitely generated r-graded S-module.

Proposition 1.5.1 *Let L be a finitely generated r-graded S-module of relevant dimension δ. Then there exists a polynomial $P_L(t_1,\dots,t_r) \in \mathbb{Q}[t_1,\dots,t_r]$ of total degree $\delta - r$ such that $H(L,i_1,\dots,i_r) = P_L(i_1,\dots,i_r)$ for $i_1 \gg d_2^1 i_2 + \dots + d_r^1 i_r$, $\dots$, $i_{r-1} \gg d_r^{r-1} i_r$, $i_r \gg 0$. Moreover,*

$$P_L(t_1,\dots,t_r) = \sum_{|\mathbf{n}|\leq\delta-r} a_{\mathbf{n}} \binom{t_1 - d_2^1 t_2 - \dots - d_r^1 t_r}{n_1} \cdots \binom{t_{r-1} - d_r^{r-1} t_r}{n_{r-1}} \binom{t_r}{n_r},$$

where $a_{\mathbf{n}} \in \mathbb{Z}$, $a_{\mathbf{n}} \geq 0$ *if* $|\mathbf{n}| = \delta - r$.

Proof. Given a finitely generated r-graded S-module L, first note that there is a chain

$$0 = L_0 \subset L_1 \subset \ldots \subset L_s = L$$

of r-graded submodules of L such that for each $i \geq 1$, $L_i/L_{i-1} \cong (S/P_i)(\mathbf{m}_i)$, where $P_i \in \operatorname{Supp} L$ is a homogeneous prime ideal and $\mathbf{m}_i \in \mathbb{Z}^r$. Indeed, we may assume $L \neq 0$. Choose $P_1 \in \operatorname{Ass} L$. Then P_1 is a homogeneous prime ideal, and there exists an r-graded submodule $L_1 \subset L$ such that $L_1 \cong (S/P_1)(\mathbf{m}_1)$. If $L_1 \neq L$, by repeating the procedure with L/L_1 we get an r-graded submodule $L_2 \subset L$ such that $L_2/L_1 \cong (S/P_2)(\mathbf{m}_2)$. Since L is noetherian, this process finishes after a finite number of steps. From this chain, we obtain

$$H(L, \mathbf{n}) = \sum_{i=1}^{s} H(S/P_i, \mathbf{n} + \mathbf{m}_i).$$

So it is enough to prove the result for the rings $T = S/P$, with P a homogeneous prime ideal. To this end, we will reduce the problem to the standard case where the result is already known.

Set $B = T_0$. Let us consider $\overline{T} \subset T$ the B-algebra generated by the homogeneous elements of T of degree $(e_1, \ldots, e_r)$ such that

$$\begin{aligned}
&e_{r-1} \geq d_r^{r-1} e_r \\
&e_{r-2} \geq d_{r-1}^{r-2} e_{r-1} + d_r^{r-2} e_r \\
&\cdots\cdots \\
&e_1 \geq d_2^1 e_2 + \ldots + d_r^1 e_r.
\end{aligned}$$

Then one has $\overline{T}_{\mathbf{n}} = T_{\mathbf{n}}$ for each $\mathbf{n} \in \mathbb{N}^r$ satisfying the inequalities before. Let us consider the morphism

$$\begin{array}{cccc}
\psi: & \mathbb{Z}^r & \longrightarrow & \mathbb{Z}^r \\
& (x_1, \ldots, x_r) & \mapsto & (x_1 - d_2^1 x_2 - \cdots - d_r^1 x_r, \ldots, x_{r-1} - d_r^{r-1} x_r, x_r)
\end{array}$$

Note that $\psi(\operatorname{supp} \overline{T}) \subset \mathbb{N}^r$, so $\overline{T}^{\psi}$ is again a $\mathbb{N}^r$-graded ring. Furthermore, we have $\operatorname{rel.dim} \overline{T}^{\psi} = \operatorname{rel.dim} \overline{T} = \operatorname{rel.dim} T = \delta$. If $\overline{T}^{\psi}$ is standard, by [HHRT, Theorem 4.1] there exists a polynomial $Q(t_1, \ldots, t_r) \in \mathbb{Q}[t_1, \ldots, t_r]$ of total degree $\delta - r$

$$Q(t_1, \ldots, t_r) = \sum_{|\mathbf{n}| \leq \delta - r} a_{\mathbf{n}} \binom{t_1}{n_1} \cdots \binom{t_r}{n_r},$$

with $a_{\mathbf{n}} \in \mathbb{Z}$, $a_{\mathbf{n}} \geq 0$ if $|\mathbf{n}| = \delta - r$ such that for $\mathbf{i} \gg 0$

$$Q(i_1, \dots, i_r) = \mathrm{length_B}[\,\overline{\mathrm{T}}^{\psi}\,]_{(\mathrm{i_1, \dots, i_r})}.$$

Then, by defining $P(t_1, \dots, t_r) = Q(t_1 - d_2^1 t_2 - \dots d_r^1 t_r, \dots, t_{r-1} - d_r^{r-1} t_r, t_r)$, let us observe that for $i_1 \gg d_2^1 i_2 + \dots + d_r^1 i_r, \dots, i_{r-1} \gg d_r^{r-1} i_r, i_r \gg 0$, we have

$$\begin{aligned} P(i_1, \dots, i_r) &= \mathrm{length_B}\ [\overline{\mathrm{T}}^{\psi}]_{(\mathrm{i_1 - d_2^1 i_2 - \dots - d_r^1 i_r, \dots, i_{r-1} - d_r^{r-1} i_r, i_r})} \\ &= \mathrm{length_B}\ [\overline{\mathrm{T}}]_{(\mathrm{i_1, \dots, i_r})} \\ &= \mathrm{length_A}\ [\mathrm{T}]_{(\mathrm{i_1, \dots, i_r})}, \end{aligned}$$

so we get the statement.

Therefore we only have to prove that $\overline{T}^{\psi}$ is standard or, equivalently, that $\overline{T}$ can be generated over B by homogeneous elements in degrees $(e_1, \dots, e_r)$ such that $e_{i+1} = \dots = e_r = 0$, $e_i = 1$, $e_{i-1} = d_i^{i-1} e_i$, $e_{i-2} = d_{i-1}^{i-2} e_{i-1} + d_i^{i-2} e_i$, …, $e_1 = d_2^1 e_2 + \dots + d_i^1 e_i$. Assume that T is generated over B by homogeneous elements $z_{11}, \dots, z_{1k_1}, \dots, z_{r1}, \dots, z_{rk_r}$ in degrees $\deg(z_{ij}) = (d_{ij}^1, \dots, d_{ij}^{i-1}, 1, 0, \dots, 0)$. Let us take a homogeneous element z in $\overline{T}$, with $\deg z = (\alpha_1, \dots, \alpha_r)$. Let j be such that $\alpha_j \neq 0$, $\alpha_{j+1} = \dots = \alpha_r = 0$ (j is 0 if $z \in B$). We are going to prove by induction on j that z can be generated over B by the homogeneous elements whose degrees satisfy the equalities before. If $j = 0$, there is nothing to prove. If $j = 1$, then $\deg z = (\alpha_1, 0, \dots, 0)$ and we can write z as a linear combination with coefficients in B of products of α_1 elements among $z_{11}, \dots, z_{1k_1}$, so the result is trivial. Assume now that $j > 1$. By forgetting the first component of the degree, we have by induction hypothesis that z can be written as a sum of terms of the type $\lambda w_1 \dots w_l$ with $\lambda \in B[z_{11}, \dots, z_{1k_1}]$, and the degree of the elements w_i satisfying the $r-1$ first equalities. Set $\deg w_j = (s_j^1, \dots, s_j^r)$, $\deg \lambda = (s, 0, \dots, 0)$. We will finish if we prove that

$$\alpha_1 \geq \sum_{j=1}^{l} (d_2^1 s_j^2 + \dots + d_r^1 s_j^r).$$

But note that $\alpha_1 \geq d_2^1 \alpha_2 + \dots + d_r^1 \alpha_r = \sum_{j=1}^{l} d_2^1 s_j^2 + \dots + d_r^1 s_j^r$. $\square$

Our next aim will be to study for a given finitely generated r-graded S-module L, the A-modules $H^i_{S_+}(L)_{\mathbf{n}}$ for $i \geq 0$, $\mathbf{n} \in \mathbb{Z}^r$. We need two previous lemmas.

Lemma 1.5.2 *Let L be a finitely generated r-graded S-module such that $(S_+)^u L = 0$ for an integer u. Then there exits $\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{Z}^r$ such that*

$$L_{\mathbf{n}} = 0,$$

for $\mathbf{n} = (n_1, \dots, n_r)$ such that $n_1 > d_2^1 n_2 + \dots + d_r^1 n_r + m_1, \dots, n_{r-1} > d_r^{r-1} n_r + m_{r-1}, n_r > m_r$.

Proof. Since there exists $u \in \mathbb{Z}$ such that $(S_+)^u L = 0$, we have $\operatorname{Supp}_+(L) = V_+(\operatorname{Ann} L) \subset V_+(S_+) = \emptyset$, so $\operatorname{rel.dim} L = r - 1$. Then the result follows from Proposition 1.5.1. $\square$

Lemma 1.5.3 *(Homogeneous Prime Avoidance) Let $P_1, \dots, P_m \in \operatorname{Proj}^r(S)$. If I is any homogeneous ideal of S such that $I \not\subset P_i$ for $i = 1, \dots, m$, then there is a homogeneous element a such that $a \in I$, $a \notin P_1 \cup \dots \cup P_m$.*

Proof. We may assume that $P_j \not\subset P_i$ for $i \neq j$, so for a given i, we have that for any $j \neq i$ there exists a homogeneous element $p_{ij} \in P_j$, $p_{ij} \notin P_i$. Then $p_i = \prod_{j \neq i} p_{ij}$ satisfies that $p_i \notin P_i$, but $p_i \in P_j$ for all $j \neq i$. Next we may take homogeneous elements $a_i \in I$, $a_i \notin P_i$ for $i = 1, \dots, m$. Set $\deg a_i p_i = (\alpha_{i1}, \dots, \alpha_{ir})$. Since $S_+ \not\subset P_i$, there exists an element of the type $x_{1j_1} \dots x_{rj_r} \notin P_i$. So multiplying each $a_i p_i$ by a power of the corresponding x_{rj_r} we can assume that $\alpha_{1r} = \dots = \alpha_{mr} = \alpha_r$. Then, multiplying by suitable powers of each $x_{r-1, j_{r-1}}$ we may also assume that $\alpha_{1,r-1} = \dots = \alpha_{m,r-1} = \alpha_{r-1}$. By repeating this procedure as many times as necessary, we can assume at the end that $\deg(a_1 p_1) = \dots = \deg(a_m p_m) = (\alpha_1, \dots, \alpha_r)$. Now $a = a_1 p_1 + \dots + a_m p_m$ is homogeneous and $a \in I$, $a \notin P_1 \cup \dots \cup P_m$. $\square$

Now we are ready to prove that if L is a finitely generated r-graded S-module, then the A-modules $H^i_{S_+}(L)_{\mathbf{n}}$ are finitely generated for all $\mathbf{n} \in \mathbb{Z}^r$, $i \geq 0$, and vanish for all sufficiently large $\mathbf{n}$. Here, the artinian assumption on A is not necessary. In the graded case, this is a classical result due to J.P. Serre.

Proposition 1.5.4 *Let L be a finitely generated r-graded S-module. Then*

(i) For all $i \geq 0$, $\mathbf{n} \in \mathbb{Z}^r$, the A-module $H^i_{S_+}(L)_{\mathbf{n}}$ is finitely generated.

(ii) There exits $\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{Z}^r$ such that $H^i_{S_+}(L)_{\mathbf{n}} = 0$ for all $i \geq 0$, $\mathbf{n} = (n_1, \dots, n_r)$ such that $n_1 > d_2^1 n_2 + \dots + d_r^1 n_r + m_1, \dots, n_{r-1} > d_r^{r-1} n_r + m_{r-1}, n_r > m_r$.

Proof. We will follow the same lines as the proof of the graded version in [BroSha, Proposition 15.1.5]. We will prove by induction on i that $H^i_{S_+}(L)_{\mathbf{n}}$ is a finitely generated A-module for all $\mathbf{n} \in \mathbb{Z}^r$, and that it is zero for all sufficiently large values of $\mathbf{n}$. This proves the statement because $H^i_{S_+}(L) = 0$ for all i greater than the minimal number of generators of S_+.

Assume $i = 0$. Then $H^0_{S_+}(L)$ is a finitely generated r-graded S-module since it is a submodule of L, and so $H^0_{S_+}(L)_{\mathbf{n}}$ is a finitely generated A-module and there exists $u \in \mathbb{N}$ such that $(S_+)^u H^0_{S_+}(L) = 0$. Then, according to Lemma 1.5.2 there exists $\mathbf{m} \in \mathbb{Z}^r$ such that $H^0_{S_+}(L)_{\mathbf{n}} = 0$ for $n_1 > d^1_2 n_2 + \ldots + d^1_r n_r + m_1$, …, $n_{r-1} > d^{r-1}_r n_r + m_{r-1}$, $n_r > m_r$.

Now let us assume $i > 0$. From the r-graded isomorphism $H^i_{S_+}(L) \cong H^i_{S_+}(L/H^0_{S_+}(L))$, we may replace L by $L/H^0_{S_+}(L)$ and then assume that $H^0_{S_+}(L) = 0$. Then $S_+ \not\subset P$ for all $P \in \operatorname{Ass}(L)$, and so by the Prime Avoidance Lemma there exists a homogeneous element $x \in S_+$ of degree $\mathbf{k} = (k_1, \ldots, k_r)$ such that $x \notin P$ for all $P \in \operatorname{Ass}(L)$. Looking at the proof of the Prime Avoidance Lemma, notice that we can choose x such that $\mathbf{k}$ satisfies $k_1 > d^1_2 k_2 + \cdots + d^1_r k_r$, …, $k_{r-1} > d^{r-1}_r k_r$. Then we get an r-graded exact sequence

$$0 \to L(-\mathbf{k}) \xrightarrow{\cdot x} L \to L/xL \to 0,$$

which induces for all $\mathbf{n} \in \mathbb{Z}^r$ the exact sequence of A-modules

$$H^{i-1}_{S_+}(L/xL)_{\mathbf{n}} \to H^i_{S_+}(L)_{\mathbf{n}-\mathbf{k}} \xrightarrow{\cdot x} H^i_{S_+}(L)_{\mathbf{n}} \quad .$$

By the induction hypothesis, there exists $\overline{\mathbf{m}} \in \mathbb{Z}^r$ such that $H^{i-1}_{S_+}(L/xL)_{\mathbf{n}} = 0$ for all $\mathbf{n} = (n_1, \ldots, n_r)$ such that $n_1 > d^1_2 n_2 + \ldots + d^1_r n_r + \overline{m}_1$, …, $n_{r-1} > d^{r-1}_r n_r + \overline{m}_{r-1}$, $n_r > \overline{m}_r$. Now let $\mathbf{n}$ verifying these inequalities. Then note that for any $s \geq 1$, $\mathbf{n} + s\mathbf{k}$ also satifies them, and so we have exact sequences

$$0 \to H^i_{S_+}(L)_{\mathbf{n}-\mathbf{k}} \xrightarrow{\cdot x^s} H^i_{S_+}(L)_{\mathbf{n}+(s-1)\mathbf{k}} \quad .$$

Since $H^i_{S_+}(L)$ is S_+-torsion and $x \in S_+$, we have $H^i_{S_+}(L)_{\mathbf{n}-\mathbf{k}} = 0$. Therefore, by taking $\mathbf{m} = \overline{\mathbf{m}} - \mathbf{k}$, we obtain $H^i_{S_+}(L)_{\mathbf{n}} = 0$ for all $\mathbf{n}$ such that $n_1 > d^1_2 n_2 + \ldots + d^1_r n_r + m_1$, …, $n_{r-1} > d^{r-1}_r n_r + m_{r-1}$, $n_r > m_r$.

Now let us fix $\mathbf{n} \in \mathbb{Z}^r$. If $n_1 > d^1_2 n_2 + \ldots + d^1_r n_r + m_1$, …, $n_{r-1} > d^{r-1}_r n_r + m_{r-1}$, $n_r > m_r$, we have that $H^i_{S_+}(L)_{\mathbf{n}} = 0$, and so it is a finitely generated A-module. Otherwise, let us take $y \in S_+$ such that $y \notin \bigcup_{P \in \operatorname{Ass}(L)} P$ with degree $\mathbf{l} = (l_1, \ldots, l_r)$ such that $\mathbf{n} + \mathbf{l}$ satisfies the previous inequalities

(we can find such a y by the Prime Avoidance Lemma). Then we have the graded exact sequence

$$H^{i-1}_{S_+}(L/yL)_{\mathbf{n+1}} \to H^i_{S_+}(L)_{\mathbf{n}} \xrightarrow{\cdot y} H^i_{S_+}(L)_{\mathbf{n+1}} = 0,$$

and from the induction hypothesis we also have that $H^{i-1}_{S_+}(L/yL)_{\mathbf{n+1}}$ is a finitely generated A-module, and so $H^i_{S_+}(L)_{\mathbf{n}}$. $\square$

We have already shown that the Hilbert function of any finitely generated r-graded S-module is a polynomial function for large $\mathbf{n}$. Our next result precises the difference between the Hilbert function and the Hilbert polynomial for any $\mathbf{n}$.

Proposition 1.5.5 *Let L be a finitely generated r-graded S-module. Then for all $\mathbf{n} \in \mathbb{Z}^r$*

$$H(L,\mathbf{n}) - P_L(\mathbf{n}) = \sum_q (-1)^q \operatorname{length}_A(H^q_{S_+}(L)_{\mathbf{n}}).$$

Proof. We will follow the proof of the graded version from [BH1, Theorem 4.3.5]. For an arbitrary finitely generated r-graded S-module L, let us define the series

$$H'_L(u_1,\dots,u_r) = \textstyle\sum_{\mathbf{n}\in\mathbb{Z}^r}(H(L,\mathbf{n}) - P_L(\mathbf{n}))u^{\mathbf{n}}$$

$$H''_L(u_1,\dots,u_r) = \textstyle\sum_{\mathbf{n}\in\mathbb{Z}^r}\big(\sum_q (-1)^q \operatorname{length}_A(H^q_{S_+}(L)_{\mathbf{n}})\big)u^{\mathbf{n}}.$$

We will prove the statement by induction on $\delta = \operatorname{rel.dim} L$. If $\delta = r-1$, then $\operatorname{Supp}_+ L = \emptyset$, and so there exists m such that $S_+^m \subset \operatorname{Ann}(L)$. Therefore $H^0_{S_+}(L) = L$, and hence the result is trivial. Assume now $\delta \geq r$, and let us consider $\overline{L} = L/H^0_{S_+}(L)$. Since $H^0_{S_+}(L)$ is a finitely generated r-graded S-module which is vanished by some power of S_+, there are integers $i_1,\dots,i_r$ such that $H^0_{S_+}(L)_{\mathbf{n}} = 0$ for $n_1 > d_2^1 n_2 + \ldots + d_r^1 n_r + i_1$, …, $n_{r-1} > d_r^{r-1} n_r + i_{r-1}$, $n_r > i_r$. So we have $P_L(\mathbf{t}) = P_{\overline{L}}(\mathbf{t})$, and it is enough to prove the result for $\overline{L}$ because then, for all $\mathbf{n} = (n_1,\dots,n_r)$

$$\begin{aligned} H(L,\mathbf{n}) - P_L(\mathbf{n}) &= H(\overline{L},\mathbf{n}) + \operatorname{length}_A(H^0_{S_+}(L)_{\mathbf{n}}) - P_{\overline{L}}(\mathbf{n}) \\ &= \textstyle\sum_q (-1)^q \operatorname{length}_A(H^q_{S_+}(\overline{L})_{\mathbf{n}}) + \operatorname{length}_A(H^0_{S_+}(L)_{\mathbf{n}}) \\ &= \textstyle\sum_q (-1)^q \operatorname{length}_A(H^q_{S_+}(L)_{\mathbf{n}}). \end{aligned}$$

So let us assume $H^0_{S_+}(L) = 0$. Then $S_+ \not\subset P$ for all $P \in \operatorname{Ass}(L)$, and so by the Prime Avoidance Lemma there exists a homogeneous element $x \in S_+$ of

degree $\mathbf{k} = (k_1, \ldots, k_r)$ such that $x \notin P$ for all $P \in \mathrm{Ass}(L)$. Then we have the r-graded exact sequence

$$0 \to L(-\mathbf{k}) \to L \to L/xL \to 0,$$

with $\mathrm{rel.dim}\, L/xL < \mathrm{rel.dim}\, L$. Note that $H(L/xL, \mathbf{n}) = H(L, \mathbf{n}) - H(L, \mathbf{n}-\mathbf{k})$ for all $\mathbf{n}$, and so $P_{L/xL}(\mathbf{t}) = P_L(\mathbf{t}) - P_L(\mathbf{t} - \mathbf{k})$. We conclude $H'_{L/xL}(\mathbf{u}) = (1 - \mathbf{u}^{\mathbf{k}})H'_L(\mathbf{u})$. From the long exact sequence of local cohomology, we also get $H''_{L/xL}(\mathbf{u}) = (1-\mathbf{u}^{\mathbf{k}})H''_L(\mathbf{u})$. By the induction hypothesis, we have $H'_{L/xL}(\mathbf{u}) = H''_{L/xL}(\mathbf{u})$ and so $H'_L(\mathbf{u}) = H''_L(\mathbf{u})$. $\square$

Chapter 2

The diagonals of a bigraded module

Throughout this chapter we will study in more detail the diagonal functor in the category of bigraded S-modules, where $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ is the polynomial ring in $n+r$ variables with the bigrading given by $\deg(X_i) = (1, 0)$, $\deg(Y_j) = (d_j, 1)$, and $d_1, \ldots, d_r \geq 0$. This category includes any standard bigraded k-algebra, by taking $d_1 = \ldots = d_r = 0$, as well as the Rees ring and the form ring of a homogeneous ideal in a graded k-algebra, when those rings are endowed with an appropiate bigrading (see Section 2.3).

For a given c, e positive integers, let Δ be the (c, e)-diagonal of $\mathbb{Z}^2$. Our purpose is to study the exact functor $(\ )_\Delta : M^2(S) \to M^1(S_\Delta)$ (see Chapter 1, Section 4). We are mainly interested in studying how the arithmetic properties of a bigraded S-module L and its diagonals L_Δ are related. Most of these properties, like the Cohen-Macaulayness or the Gorenstein property, can be characterized by means of the local cohomology modules. So it would be very useful to relate the local cohomology modules of L with the local cohomology modules of its diagonals. This has been done by A. Conca et al. in [CHTV] from the study of the bigraded minimal free resolution of L over S, after developing a theory of generalized Segre products of bigraded algebras. In Section 2.1 we are going to present their results by a different and somewhat easier approach. In addition, this approach will provide more detailed information about several problems concerning to the behaviour of the local cohomology when taking diagonals.

In Section 2.2 we focus our study on standard bigraded k-algebras. For

a such k-algebra R, let $\mathcal{R}_1 = \bigoplus_{i\in\mathbb{N}} R_{(i,0)}$, $\mathcal{R}_2 = \bigoplus_{j\in\mathbb{N}} R_{(0,j)}$. In this case, we give a characterization for R to have a good resolution in terms of the a_*-invariants of $\mathcal{R}_1$ and $\mathcal{R}_2$ which, in particular, provides a criterion for the Cohen-Macaulayness of its diagonals. We also find necessary and sufficient conditions on the local cohomology of $\mathcal{R}_1$ and $\mathcal{R}_2$ for the existence of Cohen-Macaulay diagonals of R, whenever R is Cohen-Macaulay.

Given a homogeneous ideal I in a graded k-algebra A, the Rees algebra $R_A(I) = \bigoplus_{n\geq 0} I^n$ of I can be endowed with the bigrading $R_A(I)_{(i,j)} = (I^j)_i$. The last section of the chapter is devoted to study the diagonals of the Rees algebra. In the case where A is the polynomial ring, we will show that if the Rees algebra is Cohen-Macaulay then there exists some diagonal with this property, thus proving a conjecture stated in [CHTV]. Furthermore, we will give necessary and sufficient conditions on the ring A for the existence of a Cohen-Macaulay diagonal of a Cohen-Macaulay Rees algebra.

2.1 The diagonal functor on the category of bigraded modules

Let $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ be the polynomial ring in $n + r$ variables over a field k with the bigrading given by $\deg(X_i) = (1, 0)$, $\deg(Y_j) = (d_j, 1)$, where $d_1, \ldots, d_r \geq 0$. Set $d = \max\{d_1, \ldots, d_r\}$, $u = \sum_{j=1}^r d_j$. Let us denote by $\mathcal{M}$ the homogeneous maximal ideal of S. Note that the irrelevant ideal S_+ of S is the ideal generated by the products X_iY_j, for $i = 1, \ldots, n$, $j = 1, \ldots, r$.

Given c, e positive integers, let Δ be the (c, e)-diagonal of $\mathbb{Z}^2$. For any bigraded S-module L, let us recall that the diagonal of L along Δ is defined as $L_\Delta := \bigoplus_{s\in\mathbb{Z}} L_{(cs,es)}$, which is a graded module over the graded ring $S_\Delta := \bigoplus_{s\geq 0} S_{(cs,es)}$. Our first lemma computes the dimension of the diagonals of a finitely generated bigraded S-module.

Lemma 2.1.1 *Let L be a finitely generated bigraded S-module. For $\Delta = (c, e)$ with $c \geq de + 1$,* $\dim L_\Delta = \text{rel.dim } L - 1$.

Proof. The proof follows the same lines as the one given for the bigraded standard case by A. Simis et al. in [STV, Proposition 2.3]. Set $\delta = \text{rel.dim}\, L$.

According to Proposition 1.5.1, there is a polynomial $P(s,t) \in \mathbb{Q}[s,t]$ of total degree $\delta - 2$ of the type

$$P(s,t) = \sum_{k+l \leq \delta-2} a_{kl} \binom{s-dt}{k} \binom{t}{l},$$

with $a_{kl} \geq 0$ for any k,l verifying $k+l = \delta - 2$ such that for $i \gg dj$, $j \gg 0$, $P(i,j) = \dim_k L_{(i,j)}$. For any $c \geq de+1$, let us consider the polynomial $Q(u) = P(cu, eu) \in \mathbb{Q}[u]$. Then $Q(u) = \dim_k L_{(cu,eu)} = \dim_k (L_\Delta)_u$ for u large enough and $\deg Q(u) = \delta - 2$. Therefore $\dim L_\Delta = \delta - 1$. $\square$

From now on in the chapter we will always consider diagonals $\Delta = (c,e)$ with $c \geq de+1$. The next two propositions are inspired in some results and techniques used by E. Hyry in [Hy]. The first one shows how the local cohomology modules of L with respect to S_+ are related to the local cohomology modules of L_Δ with respect to $\mathcal{M}_\Delta$.

Proposition 2.1.2 *Let L be a finitely generated bigraded S-module. Then there are graded isomorphisms*

$$H^q_{S_+}(L)_\Delta \cong H^q_{\mathcal{M}_\Delta}(L_\Delta) \ \ ,\forall q.$$

Proof. Let $\mathcal{N}$ be the ideal of S generated by $\mathcal{M}_\Delta$. Observe that $\sqrt{S_+} = \sqrt{\mathcal{N}}$, so we immediately get a bigraded isomorphism $H^q_{S_+}(L) \cong H^q_{\mathcal{N}}(L)$, $\forall q \geq 0$. Denoting by $g_1, \ldots, g_s$ a k-basis of $S_{(c,e)}$, we have that $\mathcal{N}$ can be generated by $g_1, \ldots, g_s$. So we may compute the local cohomology modules of L with respect to $\mathcal{N}$ from the Čech complex built up from these elements

$$\mathbf{C}^{\cdot} : 0 \to C^0 \to C^1 \to \ldots \to C^s \to 0,$$

$$C^t = \bigoplus_{1 \leq i_1 < i_2 < \ldots < i_t \leq s} L_{g_{i_1} g_{i_2} \cdots g_{i_t}},$$

with the differentation $d^t : C^t \to C^{t+1}$ defined on the component

$$L_{g_{i_1} g_{i_2} \cdots g_{i_t}} \longrightarrow L_{g_{j_1} g_{j_2} \cdots g_{j_t} g_{j_{t+1}}}$$

to be the homomorphism $(-1)^{m-1}\mathrm{nat} : L_{g_{i_1} g_{i_2} \cdots g_{i_t}} \to (L_{g_{i_1} g_{i_2} \cdots g_{i_t}})_{g_{j_m}}$ if $\{i_1, \ldots, i_t\} = \{j_1, \ldots, \widehat{j_m}, \ldots, j_{t+1}\}$, and 0 otherwise. Then $H^q_{\mathcal{N}}(L) \cong H^q(\mathbf{C}^{\cdot})$. We can also consider the Čech complex associated to L_Δ built up from $g_1, \ldots, g_s$

$$\mathbf{D}^{\cdot} : 0 \to D^0 \to D^1 \to \ldots \to D^s \to 0,$$

$$D^t = \bigoplus_{1\le i_1<i_2<\ldots<i_t\le s} (L_\Delta)_{g_{i_1}g_{i_2}\cdots g_{i_t}},$$

with the differentation $\delta^t : D^t \to D^{t+1}$ defined on the component

$$(L_\Delta)_{g_{i_1}g_{i_2}\cdots g_{i_t}} \longrightarrow (L_\Delta)_{g_{j_1}g_{j_2}\cdots g_{j_t}g_{j_{t+1}}}$$

to be the homomorphism $(-1)^{m-1}\text{nat} : (L_\Delta)_{g_{i_1}g_{i_2}\cdots g_{i_t}} \to ((L_\Delta)_{g_{i_1}g_{i_2}\cdots g_{i_t}})_{g_{j_m}}$ if $\{i_1,\ldots,i_t\} = \{j_1,\ldots,\widehat{j_m},\ldots,j_{t+1}\}$, and 0 otherwise. Then $H^q_{\mathcal{M}_\Delta}(L_\Delta) \cong H^q(\mathbf{D}^\cdot)$. Note that $d^t|D^t = \delta^t$, so we have $(\operatorname{Ker} d^t)_\Delta = \operatorname{Ker}\delta^t$, $(\operatorname{Im} d^t)_\Delta = \operatorname{Im}\delta^t$. Therefore we may conclude $H^q_{\mathcal{N}}(L)_\Delta \cong H^q_{\mathcal{M}_\Delta}(L_\Delta)$. □

Now, let S_1, S_2 be the bigraded subalgebras of S defined by $S_1 = k[X_1,\ldots,X_n]$, $S_2 = k[Y_1,\ldots,Y_r]$, and note that the ideals $\mathfrak{m}_1 = (X_1,...,X_n)$ and $\mathfrak{m}_2 = (Y_1,\ldots,Y_r)$ are the homogeneous maximal ideals of S_1 and S_2 respectively. Then let us define $\mathcal{M}_1$ to be the ideal of S generated by $\mathfrak{m}_1$ and $\mathcal{M}_2$ to be the ideal of S generated by $\mathfrak{m}_2$. Note that $\mathcal{M}_1 + \mathcal{M}_2 = \mathcal{M}$ and $\mathcal{M}_1 \cap \mathcal{M}_2 = S_+$. Therefore we have

Proposition 2.1.3 *Let L be a finitely generated bigraded S-module. There is a natural graded exact sequence*

$$\ldots \to H^q_{\mathcal{M}}(L)_\Delta \to H^q_{\mathcal{M}_1}(L)_\Delta \oplus H^q_{\mathcal{M}_2}(L)_\Delta \to H^q_{\mathcal{M}_\Delta}(L_\Delta) \xrightarrow{\varphi^q_L} H^{q+1}_{\mathcal{M}}(L)_\Delta \to \ldots$$

Proof. We get the result by applying the diagonal functor to the Mayer-Vietoris sequence associated to $\mathcal{M}_1$, $\mathcal{M}_2$ and by then using Proposition 2.1.2. □

As a first consequence we may recover the following result by A. Conca et al. in [CHTV].

Corollary 2.1.4 *[CHTV, Theorem 3.6] Let L be a finitely generated bigraded S-module. For all $q \ge 0$, there exists a canonical graded homomorphism*

$$\varphi^q_L : H^q_{\mathcal{M}_\Delta}(L_\Delta) \to H^{q+1}_{\mathcal{M}}(L)_\Delta,$$

which is an isomorphism for $q > \max\{n, r\}$.

Proof. Since $\mathcal{M}_1$ is generated by n elements, we have that $H^q_{\mathcal{M}_1}(L) = 0$ for any $q > n$. Similarly, $H^q_{\mathcal{M}_2}(L) = 0$ for any $q > r$. Now, the corollary follows from Proposition 2.1.3. □

Moreover, let us also notice that Proposition 2.1.3 precises the obstructions for φ^q_L to be isomorphism. Denote by $[\varphi^q_L]_s : H^q_{\mathcal{M}_\Delta}(L_\Delta)_s \to H^{q+1}_{\mathcal{M}}(L)_{(cs,es)}$ the component of degree s of the map φ^q_L. Then we have

Corollary 2.1.5 *Let L be a finitely generated bigraded S-module. For a given $s \in \mathbb{Z}$, the following are equivalent*

(i) $[\varphi_L^q]_s$ *is an isomorphism, for all* $q \geq 0$.

(ii) $H^q_{\mathcal{M}_1}(L)_{(cs,es)} = H^q_{\mathcal{M}_2}(L)_{(cs,es)} = 0$, *for all* $q \geq 0$.

In particular, φ_L^q *is an isomorphism for all* $q \geq 0$ *if and only if* $H^q_{\mathcal{M}_1}(L)_\Delta = H^q_{\mathcal{M}_2}(L)_\Delta = 0$ *for all* $q \geq 0$.

Therefore, the obstructions for the maps φ_L^q to be isomorphisms are located in the vanishing of the local cohomology modules with respect to $\mathcal{M}_1$ and $\mathcal{M}_2$. So our next goal will be to study these local cohomology modules. For that, let us consider

$$0 \to D_t \to \cdots \to D_1 \to D_0 \to L \to 0$$

the $\mathbb{Z}^2$-graded minimal free resolution of L over S. For every p, D_p is a finite direct sum of S-modules of the type $S(a,b)$. If we apply the diagonal functor to this resolution, we get a resolution of L_Δ by means of the modules $S(a,b)_\Delta$. Let us begin by studying the local cohomology modules of the bigraded S-modules obtained by shifting S with degree (a,b).

First, let us fix some notations. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$, and $\beta = (\beta_1, \ldots, \beta_r) \in \mathbb{Z}^r$, we write X^α for the monomial $X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ and Y^β for the monomial $Y_1^{\beta_1} \cdots Y_r^{\beta_r}$. Note that $\deg(X^\alpha) = (\sum_{i=1}^n \alpha_i, 0)$, $\deg(Y^\beta) = (\sum_{i=1}^r d_i\beta_i, \sum_{i=1}^r \beta_i)$. We will write $\alpha < 0$ (or $\alpha \geq 0$) if all the components of α satisfy this condition, and the same for β. Then we have:

Proposition 2.1.6 *Let $a, b \in \mathbb{Z}$.*

(i)

$$H^q_{\mathcal{M}_1}(S(a,b)) = \begin{cases} 0 & \text{if } q \neq n \\ (\bigoplus_{\alpha<0,\beta\geq 0} kX^\alpha Y^\beta)(a,b) & \text{if } q = n \end{cases}$$

(ii)

$$H^q_{\mathcal{M}_2}(S(a,b)) = \begin{cases} 0 & \text{if } q \neq r \\ (\bigoplus_{\alpha\geq 0,\beta< 0} kX^\alpha Y^\beta)(a,b) & \text{if } q = r \end{cases}$$

Proof. Since $S(a,b)$ is a free S_1-module with basis the monomials in the variables $Y_1, \ldots, Y_r$, we have that $H^q_{\mathcal{M}_1}(S(a,b)) = 0$ for all $q \neq n$, and $H^n_{\mathcal{M}_1}(S(a,b)) = (\bigoplus_{\beta \geq 0} H^n_{\mathfrak{m}_1}(S_1)Y^\beta)(a,b) = (\bigoplus_{\alpha<0,\beta\geq 0} kX^\alpha Y^\beta)(a,b)$. By

taking into account that $S(a,b)$ is a free S_2-module with basis the monomials in the variables $X_1,\dots,X_n$, we also get $H^q_{\mathcal{M}_2}(S(a,b))=0$ for all $q\neq r$, and $H^r_{\mathcal{M}_2}(S(a,b))=(\bigoplus_{\alpha\geq 0} H^r_{\mathfrak{m}_2}(S_2)X^\alpha)(a,b)=(\bigoplus_{\alpha\geq 0,\beta<0} kX^\alpha Y^\beta)(a,b)$. $\square$

Corollary 2.1.7 *Let $a,b\in\mathbb{Z}$.*

(i)

$$\operatorname{supp}(\,H^q_{\mathcal{M}_1}(S(a,b))_\Delta\,)=\begin{cases}\emptyset & \text{if } q\neq n\\ \{\,s\in\mathbb{Z}\mid \frac{-b}{e}\leq s\leq \frac{bd-a-n}{c-ed}\} & \text{if } q=n\end{cases}$$

(ii)

$$\operatorname{supp}(\,H^q_{\mathcal{M}_2}(S(a,b))_\Delta\,)=\begin{cases}\emptyset & \text{if } q\neq r\\ \{\,s\in\mathbb{Z}\mid \frac{(b+r)d-u-a}{c-ed}\leq s\leq \frac{-b-r}{e}\} & \text{if } q=r\end{cases}$$

Proof. From Proposition 2.1.6, a straightforward computation gives the support by taking into account that a monomial $X^\alpha Y^\beta$ in $H^n_{\mathcal{M}_1}(S(a,b))$ has degree (p,q) with $p=\sum_{i=1}^n \alpha_i+\sum_{j=1}^r d_j\beta_j-a$ and $q=\sum_{j=1}^r\beta_j-b$. Similarly one gets (ii). $\square$

For a real number x, let us denote by $[x]=\max\{n\in\mathbb{Z}\mid n\leq x\}$ the integral part of x. The following corollary gives necessary and sufficient numerical conditions for $S(a,b)_\Delta$ to be Cohen-Macaulay in terms of the diagonal Δ and the shift (a,b). In particular, notice that S_Δ is Cohen-Macaulay for any Δ.

Corollary 2.1.8 *[CHTV, Proposition 3.4] Assume $n,r\geq 2$. For any $a,b\in\mathbb{Z}$, $S(a,b)_\Delta$ is a Cohen-Macaulay S_Δ-module if and only if $[\frac{bd-a-n}{c-ed}]<\frac{-b}{e}$ and $[\frac{-b-r}{e}]<\frac{(b+r)d-u-a}{c-ed}$.*

Proof. Since S is a domain, we have that $\operatorname{rel.dim} S(a,b)=\operatorname{rel.dim} S=\dim S=n+r$, and so $\dim S(a,b)_\Delta=n+r-1$ by Lemma 2.1.1. Therefore, $S(a,b)_\Delta$ is Cohen-Macaulay if and only if $H^q_{\mathcal{M}_\Delta}(S(a,b)_\Delta)=0$ for any $q<n+r-1$. By Proposition 2.1.3, note that for $q<n+r-1$ we have that

$$H^q_{\mathcal{M}_\Delta}(S(a,b)_\Delta)\cong H^q_{\mathcal{M}_1}(S(a,b))_\Delta\oplus H^q_{\mathcal{M}_2}(S(a,b))_\Delta.$$

Since $n,r\geq 2$, we get $n+r-2\geq n,r$, and then the result follows from Corollary 2.1.7. $\square$

Remark 2.1.9 Note that if $n=r=1$, the proof above shows that $S(a,b)_\Delta$ is always Cohen-Macaulay. In the case where $n\geq 2$, $r=1$, we get that $S(a,b)_\Delta$ is Cohen-Macaulay if and only if $[\frac{-b-r}{e}]<\frac{(b+r)d-u-a}{c-ed}$, while if $n=1$, $r\geq 2$, $S(a,b)_\Delta$ is Cohen-Macaulay if and only if $[\frac{bd-a-n}{c-ed}]<\frac{-b}{e}$.

For simplicity, from now on we will assume $n, r \geq 2$. Now, let L be a finitely generated bigraded S-module. For any $p \geq 0$, let us denote by $\Omega_{p,L}$ the set of shifts (a, b) which appear in the place p of its bigraded minimal free resolution, and Ω_L the union of all these sets. Often we will write Ω_p, Ω if there is not danger of confusion with respect to the module L. The next result relates the local cohomology of the diagonals L_Δ of L to the local cohomology of the diagonals $S(a,b)_\Delta$ of the modules $S(a,b)$ which arise in its minimal free resolution.

Proposition 2.1.10 *Let L be a finitely generated bigraded S-module. Then*

(i) If $H^q_{\mathcal{M}_1}(L)_{(cs,es)} \neq 0$, then there exists a shift $(a,b) \in \Omega_{n-q,L}$ such that $H^n_{\mathcal{M}_1}(S(a,b))_{(cs,es)} \neq 0$, and so $\frac{-b}{e} \leq s \leq \frac{bd-a-n}{c-ed}$.

(ii) If $H^q_{\mathcal{M}_2}(L)_{(cs,es)} \neq 0$, then there exists a shift $(a,b) \in \Omega_{r-q,L}$ such that $H^r_{\mathcal{M}_2}(S(a,b))_{(cs,es)} \neq 0$, and so $\frac{(b+r)d-u-a}{c-ed} \leq s \leq \frac{-b-r}{e}$.

Proof. To prove (i), let $0 \to D_t \to \ldots \to D_0 \to L \to 0$ be the bigraded minimal free resolution of L over S. By considering $C_p = \operatorname{Coker}(D_{p+1} \to D_p)$ for $p \geq 0$, this yields the short exact sequences

$$0 \to C_{p+1} \to D_p \to C_p \to 0, \ \ \forall p \geq 0.$$

If $H^q_{\mathcal{M}_1}(L) \neq 0$, then $q \leq n$ because $\mathcal{M}_1$ is generated by n elements. In the case $q = n$, from the short exact sequence $0 \to C_1 \to D_0 \to L \to 0$, we obtain a bigraded epimorphism $H^n_{\mathcal{M}_1}(D_0) \to H^n_{\mathcal{M}_1}(L)$. Therefore, if $H^n_{\mathcal{M}_1}(L)_{(cs,es)} \neq 0$ then $H^n_{\mathcal{M}_1}(D_0)_{(cs,es)} \neq 0$, so by Corollary 2.1.7 there exists a shift $(a,b) \in \Omega_0$ such that $\frac{-b}{e} \leq s \leq \frac{bd-a-n}{c-ed}$. If $q < n$, since $H^v_{\mathcal{M}_1}(D_p) = 0$ for any $v \neq n$, we have bigraded isomorphisms

$$H^q_{\mathcal{M}_1}(L) \cong H^{q+1}_{\mathcal{M}_1}(C_1) \cong H^{q+2}_{\mathcal{M}_1}(C_2) \cong \ldots \cong H^{n-1}_{\mathcal{M}_1}(C_{n-q-1}),$$

a bigraded monomorphism

$$0 \to H^{n-1}_{\mathcal{M}_1}(C_{n-q-1}) \to H^n_{\mathcal{M}_1}(C_{n-q}),$$

and a bigraded epimorphism

$$H^n_{\mathcal{M}_1}(D_{n-q}) \to H^n_{\mathcal{M}_1}(C_{n-q}) \to 0.$$

Therefore, $H^q_{\mathcal{M}_1}(L)_{(cs,es)} \neq 0$ implies $H^n_{\mathcal{M}_1}(D_{n-q})_{(cs,es)} \neq 0$, and we are done by Corollary 2.1.7. Similarly one can prove (ii). $\square$

Remark 2.1.11 Given a finitely generated bigraded S-module L, for each diagonal $\Delta = (c, e)$ let us consider the sets of integers

$$X_p^\Delta = \bigcup_{(a,b)\in\Omega_{p,L}} \operatorname{supp}\,(H^n_{\mathcal{M}_1}(S(a,b))_\Delta),$$

$$Y_p^\Delta = \bigcup_{(a,b)\in\Omega_{p,L}} \operatorname{supp}\,(H^r_{\mathcal{M}_2}(S(a,b))_\Delta),$$

where $X_p^\Delta = Y_p^\Delta = \emptyset$ if $p < 0$. Let $X^\Delta = \bigcup_p X_p^\Delta$, $Y^\Delta = \bigcup_p Y_p^\Delta$. Then, Proposition 2.1.10 jointly with Proposition 2.1.3 says that if $s \notin X^\Delta_{n-q} \cup Y^\Delta_{r-q}$, then $[\varphi_L^q]_s$ is a monomorphism and $[\varphi_L^{q-1}]_s$ is an epimorphism. In particular, for an integer $s \notin X^\Delta \cup Y^\Delta$ then $[\varphi_L^q]_s$ is an isomorphism for any q. (In fact, it is enough to define $X^\Delta = \bigcup_{p\leq n} X_p^\Delta$ and $Y^\Delta = \bigcup_{p\leq r} Y_p^\Delta$).

Note that the set of integers s satisfying that $\frac{-b}{e} \leq s \leq \frac{bd-a-n}{c-ed}$ and $\frac{(b+r)d-u-a}{c-ed} \leq s \leq \frac{-b-r}{e}$ is empty or it only contains the integer 0 for suitable c, e, that is, the sets X^Δ and Y^Δ are contained in $\{0\}$ for those $\Delta = (c, e)$. So we immediately get:

Corollary 2.1.12 *Let L be a finitely generated bigraded S-module. There exist positive integers e_0, α such that for any $e > e_0$, $c > de + \alpha$, we have isomorphisms $[\varphi_L^q]_s : H^q_{\mathcal{M}_\Delta}(L_\Delta)_s \to H^{q+1}_{\mathcal{M}}(L)_{(cs,es)}$ for all $q \geq 0$ and $s \neq 0$.*

Proof. It is enough to take $e_0 \geq \max\{b, -b-r : (a,b) \in \Omega_L\}$ and $\alpha \geq \max\{bd-a-n, u+a-(b+r)d : (a,b) \in \Omega_L\}$. $\square$

A similar result has been obtained in [CHTV, Lemma 3.8]. Therefore, we have that for diagonals large enough the only obstruction for the map φ_L^q to be an isomorphism is located in the component of degree 0.

Definition 2.1.13 Let L be a finitely generated bigraded S-module and let

$$0 \to D_t \to \cdots \to D_1 \to D_0 \to L \to 0$$

be the bigraded minimal free resolution of L over S. Let Δ be a diagonal. We say that the resolution is good for Δ if all the modules $(D_p)_\Delta$ are Cohen-Macaulay, that is, $X^\Delta = Y^\Delta = \emptyset$. We say that the resolution is good if there exists Δ such that the resolution is good for Δ.

From Remark 2.1.11, we immediately get that if L has a good resolution for Δ then the corresponding maps φ_L^q are isomorphisms.

Corollary 2.1.14 *Let L be a finitely generated bigraded S-module whose resolution is good for Δ. Then we have graded isomorphisms*

$$\varphi_L^q : H^q_{\mathcal{M}_\Delta}(L_\Delta) \to H^{q+1}_{\mathcal{M}}(L)_\Delta,\ \forall q \geq 0.$$

Our next goal is to study the existence of diagonals Δ for which the bigraded minimal free resolution of L is good for Δ. To this end, following [CHTV] we define:

Definition 2.1.15 We say that a property holds for $c \gg 0$ relatively to $e \gg 0$ if there exists e_0 such that for all $e > e_0$ there exists a positive integer $c(e)$ (depending on e) such that this property holds for all (c, e) with $c > c(e)$. We will often write $c \gg e \gg 0$. In fact, in the statements we will prove we could replace the condition $c \gg e \gg 0$ by the stronger one that there exist positive integers e_0, α such that the property holds for $e > e_0$, $c > de + \alpha$. For simplicity, we will keep the notation and definition of $c \gg e \gg 0$ from [CHTV].

Next result provides necessary and sufficient numerical conditions for the Cohen-Macaulayness of $S(a,b)_\Delta$ for $c \gg e \gg 0$. Namely,

Proposition 2.1.16 *[CHTV, Corollary 3.5] Let $a, b \in \mathbb{Z}$. Then $S(a,b)_\Delta$ is a Cohen-Macaulay module for $c \gg e \gg 0$ if and only if a, b satisfy one of the following conditions:*

(i) $b \leq -r$ *and* $(b+r)d - u - a > 0$,

(ii) $-r < b < 0$,

(iii) $b \geq 0$ *and* $bd - a - n < 0$.

Proof. From Proposition 2.1.3, we have that $S(a,b)_\Delta$ is Cohen-Macaulay for $c \gg e \gg 0$ if and only if $0 \notin \operatorname{supp}(H^n_{\mathcal{M}_1}(S(a,b))_\Delta) \cup \operatorname{supp}(H^r_{\mathcal{M}_2}(S(a,b))_\Delta)$ for $c \gg e \gg 0$. Then the result follows from Corollary 2.1.7. $\square$

Notice that, for a given diagonal Δ, we have that $S(a,b)_\Delta$ is Cohen-Macaulay if and only if the corresponding maps $\varphi^q_{S(a,b)}$ are isomorphisms for all $q \geq 0$. On the other hand, from the proof of Proposition 2.1.16, observe that if (a,b) does not satisfy any of the conditions above then $S(a,b)_\Delta$ is never Cohen-Macaulay. Therefore, we can not hope to extend Corollary 2.1.12 to

the component of degree 0 of the maps φ_L^q. In fact, the proof of Proposition 2.1.3 shows that $[\varphi_L^q]_0$ does not depend on the diagonal Δ.

Furthermore, note that the proof of Proposition 2.1.16 also shows that if there exists Δ such that $S(a,b)_\Delta$ is Cohen-Macaulay then $S(a,b)_\Delta$ is Cohen-Macaulay for $c \gg e \gg 0$. Therefore, if a finitely generated bigraded S-module L has a good resolution, then the resolution of L is good for diagonals $\Delta = (c,e)$ with $c \gg e \gg 0$.

Up to now we have related the vanishing of the local cohomology with respect to $\mathcal{M}_1$ and $\mathcal{M}_2$ of a bigraded S-module L with the vanishing of the local cohomology with respect to $\mathcal{M}_1$ and $\mathcal{M}_2$ of the modules $S(a,b)$ which arise in the bigraded minimal free resolution of L over S. This study has led us to get sufficient conditions on the shifts (a,b) in order to φ_L^q to be isomorphisms. In the rest of the section we shall deal with the computation of the local cohomology modules of a bigraded S-module L with respect to the ideals $\mathcal{M}_1$ and $\mathcal{M}_2$ by themselves.

In Corollary 2.1.14 we have given sufficient conditions on the shifts in Ω_L to get that the maps φ_L^q are isomorphisms for large diagonals. Next we give necessary and sufficient conditions for the maps φ_L^q to be isomorphisms in terms of the local cohomology modules of L with respect to $\mathcal{M}_1$ and $\mathcal{M}_2$. Namely,

Proposition 2.1.17 *Let L be a finitely generated bigraded S-module. Then the following are equivalent:*

(i) There exists Δ such that φ_L^q is an isomorphism for all $q \geq 0$.

(ii) For large diagonals Δ, φ_L^q is an isomorphism for all $q \geq 0$.

(iii) $H^q_{\mathcal{M}_1}(L)_{(0,0)} = H^q_{\mathcal{M}_2}(L)_{(0,0)} = 0$ for all $q \geq 0$.

For an integer e and a bigraded S-module L, let us define the graded S_1-module $L^e = \oplus_{i\in\mathbb{Z}} L_{(i,e)}$. Then we have an exact functor $(\ )^e : M^2(S) \to M^1(S_1)$. The bigraded initial degree of a bigraded S-module L is defined by $\mathbf{indeg}(L) = (\mathrm{indeg}_1(L), \mathrm{indeg}_2(L))$, where

$$\mathrm{indeg}_1(L) = \min\{i \mid \exists j \ s.t.\ L_{(i,j)} \neq 0\},$$

$$\mathrm{indeg}_2(L) = \min\{j \mid \exists i \ s.t.\ L_{(i,j)} \neq 0\}.$$

Proposition 2.1.18 *Let L be a finitely generated bigraded S-module. Then:*

(i) $H^q_{\mathcal{M}_1}(L)_{(i,j)} = H^q_{\mathfrak{m}_1}(L^j)_i$ *. In particular,* $H^q_{\mathcal{M}_1}(L)_{(i,j)} = 0$ *for* $i > a_q(L^j)$ *or* $j < \operatorname{indeg}_2(L)$.

(ii) $H^q_{\mathcal{M}_2}(L)_{(i,j)} = 0$ *for* $j > a^2_*(L)$.

Proof. As S_1-module, L is the direct sum of the modules $L^e = \oplus_i L_{(i,e)}$. Since $\mathcal{M}_1$ is the ideal of S generated by $\mathfrak{m}_1 = (X_1, \dots, X_n)$, we have that $H^q_{\mathcal{M}_1}(L) = \oplus_j H^q_{\mathfrak{m}_1}(L^j)$, and so we get (i).

Now let $0 \to D_t \to \cdots \to D_1 \to D_0 \to L \to 0$ be the bigraded minimal free resolution of L over S, where $D_p = \bigoplus_{(a,b)\in\Omega_p} S(a,b)$. By taking short exact sequences as in Proposition 2.1.10, it is just enough to prove that if $j > a^2_*(L)$ then $H^q_{\mathcal{M}_2}(S(a,b))_{(i,j)} = 0$ for any $(a,b) \in \Omega_L$ and $q \geq 0$. The case $q \neq r$ is trivial. From Proposition 2.1.6, we may deduce that $H^r_{\mathcal{M}_2}(S(a,b))_{(i,j)} = 0$ for $j > -b-r$. This finishes the proof because, according to Theorem 1.3.4, $a^2_*(L) \geq -b-r$ for any $(a,b) \in \Omega_L$. □

In the particular case $d_1 = \ldots = d_r = d$, S can be thought as a standard bigraded k-algebra by a change of grading. If we consider the morphism $\varphi(p,q) = (p-dq, q)$, observe that $\varphi(\operatorname{supp} S) \subset \mathbb{N}^2$, so S^φ is a $\mathbb{N}^2$-graded ring with $[S^\varphi]_{(p,q)} = S_{(p+dq,q)}$. Noting that $\deg(X_i) = (1,0)$ for $i = 1, \dots, n$, and $\deg(Y_j) = (0,1)$ for $j = 1, \dots, r$ as elements of S^φ, we have that S^φ is standard. For a bigraded S-module L, let us recall that the S^φ-module L^φ is the S-module L with the grading defined by $[L^\varphi]_{(p,q)} = L_{(p+dq,q)}$.

Furthermore, in this case, given an integer e we can define an exact functor $(\ )_e : M^2(S) \to M^1(S_2)$ in the following way: For any bigraded S-module L, we define L_e to be the graded S_2-module $L_e = \bigoplus_{j\in\mathbb{Z}} L_{(e+dj,j)}$. Then we have

Proposition 2.1.19 *Assume that $d_1 = \ldots = d_r = d$. For any finitely generated bigraded S-module L, we have*

(i) $H^q_{\mathcal{M}_1}(L)_{(i,j)} = 0$ *for* $i > dj + a^1_*(L^\varphi)$.

(ii) $H^q_{\mathcal{M}_2}(L)_{(i,j)} = H^q_{\mathfrak{m}_2}(L_{i-dj})_j$ *. In particular,* $H^q_{\mathcal{M}_2}(L)_{(i,j)} = 0$ *for* $j > a_q(L_{i-dj})$.

Proof. Let $0 \to D_t \to \cdots \to D_1 \to D_0 \to L \to 0$ be the bigraded minimal free resolution of L over S. Observe that

$$S(a,b)^\varphi = \bigoplus_{(i,j)} S(a,b)_{(i+dj,j)}$$

$$= \bigoplus_{(i,j)} S_{(a+i+dj,b+j)}$$
$$= \bigoplus_{(i,j)} S_{(a-db+i+d(b+j),b+j)}$$
$$= S^{\varphi}(a-db,b),$$

so in particular $S(a,b)^{\varphi}$ is a free S^{φ}-module. Therefore, by applying the exact functor $(\ )^{\varphi}$ to the resolution of L we get that

$$0 \to D_t^{\varphi} \to \ldots \to D_0^{\varphi} \to L^{\varphi} \to 0$$

is a bigraded minimal free resolution of L^{φ} over S^{φ}. Since $a^1(S^{\varphi}) = -n$, from Theorem 1.3.4 it follows that

$$a_*^1(L^{\varphi}) = \max\{\, db - a \mid (a,b) \in \Omega_L \} - n.$$

If i,j are such that $i > dj + a_*^1(L^{\varphi})$, then we have that $i > dj + db - a - n$ for any shift $(a,b) \in \Omega_L$, and so from Proposition 2.1.6 we have that $H^q_{\mathcal{M}_1}(S(a,b))_{(i,j)} = 0$ for any $q \geq 0$. By taking short exact sequences as in Proposition 2.1.18, we then obtain $H^q_{\mathcal{M}_1}(L)_{(i,j)} = 0$ for $q \geq 0$, $i > dj + a_*^1(L^{\varphi})$.

To prove (ii), note that since $d_1 = \ldots = d_r = d$ we may decompose L as the direct sum of the S_2-modules L_i. Then, by using that $\mathcal{M}_2$ is the ideal of S generated by $\mathfrak{m}_2 = (Y_1, \ldots, Y_r)$, we obtain $H^q_{\mathcal{M}_2}(L) = \bigoplus_i H^q_{\mathfrak{m}_2}(L_i)$. Noting that $\deg(Y_1) = \ldots = \deg(Y_r) = (d,1)$, we finally get $H^q_{\mathcal{M}_2}(L)_{(i,j)} = H^q_{\mathfrak{m}_2}(L_{i-dj})_j$. $\square$

2.2 Case study: Standard bigraded k-algebras

Our aim in this section is to particularize and improve for standard bigraded k-algebras several results proved in Section 2.1. So let R be a standard bigraded k-algebra generated by homogeneous elements $x_1, \ldots, x_n$, $y_1, \ldots, y_r$ in degrees $\deg(x_i) = (1,0)$, $i = 1, \ldots, n$, $\deg(y_j) = (0,1)$, $j = 1, \ldots, r$. By taking the polynomial ring $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ with the bigrading given by $\deg(X_i) = (1,0)$, $\deg(Y_j) = (0,1)$, we have that R is a finitely generated bigraded S-module in a natural way.

In this case, denote by $\mathcal{R}_1 = R^0 = \bigoplus_{i \in \mathbb{N}} R_{(i,0)}$, $\mathcal{R}_2 = R_0 = \bigoplus_{j \in \mathbb{N}} R_{(0,j)}$. Observe that $\mathcal{R}_1$ and $\mathcal{R}_2$ are graded k-algebras, and denote by $\mathfrak{m}_1$ and $\mathfrak{m}_2$ their homogeneous maximal ideals. Given $e \in \mathbb{Z}$, we may define the graded $\mathcal{R}_1$-module $R^e = \oplus_{i \in \mathbb{Z}} R_{(i,e)}$ and the graded $\mathcal{R}_2$-module $R_e = \oplus_{j \in \mathbb{Z}} R_{(e,j)}$. By a straightforward application of Proposition 2.1.3, we get

Proposition 2.2.1 *There is a natural graded exact sequence*

$$\ldots \to H^q_{\mathcal{M}}(R)_\Delta \to H^q_{\mathcal{M}_1}(R)_\Delta \oplus H^q_{\mathcal{M}_2}(R)_\Delta \to H^q_{\mathcal{M}_\Delta}(R_\Delta) \xrightarrow{\varphi^q_R} H^{q+1}_{\mathcal{M}}(R)_\Delta \to \ldots$$

In particular, given $s \in \mathbb{Z}$, the following are equivalent:

(i) $[\varphi^q_R]_s$ *is isomorphism,* $\forall q \geq 0$.

(ii) $H^q_{\mathfrak{m}_1}(R^{es})_{cs} = 0$ *and* $H^q_{\mathfrak{m}_2}(R_{cs})_{es} = 0$, $\forall q \geq 0$.

Proof. It follows from Proposition 2.1.3, Proposition 2.1.18 and Proposition 2.1.19. □

As a direct consequence of Proposition 2.2.1 we have:

Corollary 2.2.2 φ^q_R *is an isomorphism for* $q > \max\{\dim \mathcal{R}_1, \dim \mathcal{R}_2\}$.

Proof. Set $d_1 = \dim \mathcal{R}_1$, $d_2 = \dim \mathcal{R}_2$. It is enough to prove that $H^q_{\mathfrak{m}_1}(R^e) = H^q_{\mathfrak{m}_2}(R_e) = 0$ for any $e \in \mathbb{Z}$, $q > \max\{d_1, d_2\}$. But note that R^e is a graded $\mathcal{R}_1$-module, so $H^q_{\mathfrak{m}_1}(R^e) = 0$ for $q > d_1$. Similarly, $H^q_{\mathfrak{m}_2}(R_e) = 0$ for $q > d_2$ and we are done. □

From Proposition 2.2.1 we can also determine a set of integers s, depending on the diagonal Δ, for which $[\varphi^q_R]_s$ is an isomorphism for all $q \geq 0$. More explicitly,

Corollary 2.2.3 *(i)* $[\varphi^q_R]_s$ *is isomorphism for* $s < 0$.

(ii) $[\varphi^q_R]_s$ *is isomorphism for* $s > \max\{a^1_*(R)/c, a^2_*(R)/e\}$. *In particular,* $a_*(R_\Delta) \leq \max\{a^1_*(R)/c, a^2_*(R)/e\}$.

Proof. It is a direct consequence of Proposition 2.2.1, Proposition 2.1.18 and Proposition 2.1.19. □

We have shown that $[\varphi^q_R]_s$ is an isomorphism for any $s < 0$. Moreover, note that if $c > a^1_*(R)$ and $e > a^2_*(R)$, then $[\varphi^q_R]_s$ is an isomorphism for any $s > 0$. We may ensure that $[\varphi^q_R]_0$ is an isomorphism for any q if R has a good resolution. Next we study the existence of a such resolution. The following result provides a useful characterization for a standard bigraded k-algebra R to have a good resolution by means of the a_*-invariants of $\mathcal{R}_1$ and $\mathcal{R}_2$. Namely,

Proposition 2.2.4 *The following are equivalent:*

(i) R has a good resolution.

(ii) $a_*(\mathcal{R}_1) < 0$, $a_*(\mathcal{R}_2) < 0$.

Proof. Let us consider

$$\mathbf{D}_. : \ 0 \to D_t \to \cdots \to D_1 \to D_0 = S \to R \to 0$$

the bigraded minimal free resolution of R over S, where $D_p = \bigoplus_{(a,b)\in\Omega_p} S(a,b)$. Note that a, b are non-positive integers. So, by Proposition 2.1.16 the resolution is good if and only if all the shifts (a,b) satisfy one of the three following conditions

(i) $-r < b < 0$.

(ii) $b = 0$ and $-n < a$.

(iii) $b \leq -r$ and $a < 0$.

It is not hard to check that these conditions are equivalent to that for any shift $(a,0) \in \Omega_R$ we have $a > -n$, and for any shift $(0,b) \in \Omega_R$ we have $b > -r$. Observe that

$$S(a,b)^0 = \bigoplus_j S_{(a+j,b)} = \begin{cases} 0 & \text{if } b < 0 \\ S_1(a) & \text{if } b = 0 \end{cases}$$

So by applying the functor $(\ \ )^0$ to the resolution of R we obtain a graded free resolution of $\mathcal{R}_1$ over S_1:

$$\mathbf{F}_. : \ 0 \to F_t \to \cdots \to F_1 \to F_0 = S_1 \to \mathcal{R}_1 \to 0\ ,$$

where $F_p = (D_p)^0 = \bigoplus_{a\in\gamma_p} S_1(a)$, and $\gamma_p = \{a \in \mathbb{Z} : (a,0) \in \Omega_p\}$ ($F_p = 0$ if $\gamma_p = \emptyset$). Furthermore, we have that $\operatorname{Im}(F_p) \subset \mathfrak{m}_1 F_{p-1}$ for all $p = 1, ..., t$. Hence this resolution is in fact the graded minimal free resolution of $\mathcal{R}_1$ over S_1. Then we can use Theorem 1.3.4 to compute $a_*(\mathcal{R}_1)$:

$$a_*(\mathcal{R}_1) = \max\{-a \mid a \in \cup_p \gamma_p\} + a(S_1) =$$

$$= \max\{-a \mid (a,0) \in \Omega_R\} - n.$$

Therefore, any shift $(a,0) \in \Omega_R$ satisfies $a > -n$ if and only if $a_*(\mathcal{R}_1) < 0$. Similarly, any shift $(0,b) \in \Omega_R$ satisfies $b > -r$ if and only if $a_*(\mathcal{R}_2) < 0$. $\square$

As an immediate consequence we get a criterion for the existence of Cohen-Macaulay diagonals of a standard bigraded k-algebra which extends [CHTV, Corollary 3.12]. More explicitly,

Corollary 2.2.5 *Let R be a standard bigraded k-algebra with $a_*(\mathcal{R}_1) < 0$, $a_*(\mathcal{R}_2) < 0$. Then* $\operatorname{depth} R_\Delta \geq \operatorname{depth} R - 1$ *for large Δ. In particular, if R is Cohen-Macaulay, then so R_Δ for large Δ.*

For a standard bigraded ring R defined over a local ring with $a^1(R), a^2(R) < 0$, it has been shown in [Hy, Theorem 2.5] that if R is Cohen-Macaulay, then its $(1,1)$-diagonal inherits this property. This result can be extended to any diagonal of a standard bigraded k-algebra.

Proposition 2.2.6 *Let R be a standard bigraded Cohen-Macaulay k-algebra with $a^1(R), a^2(R) < 0$. Then R_Δ is Cohen-Macaulay for any diagonal Δ.*

Proof. The bigraded standard k-algebra R has a presentation as a quotient of the polynomial ring $S = k[X_1, \dots, X_n, Y_1, \dots, Y_r]$ bigraded by $\deg(X_i) = (1,0)$, $\deg(Y_j) = (0,1)$. According to Theorem 1.3.4, for any shift $(a,b) \in \Omega_R$ we have that

$$0 \leq -a \leq a^1(R) - a^1(S) < n$$

$$0 \leq -b \leq a^2(R) - a^2(S) < r.$$

Then note that for any diagonal $\Delta = (c,e)$ with $c, e > 0$,

$$X^\Delta = \bigcup_{(a,b)\in\Omega_R} \{\, s \in \mathbb{Z} \mid \frac{-b}{e} \leq s \leq \frac{-a-n}{c} \,\} = \emptyset$$

$$Y^\Delta = \bigcup_{(a,b)\in\Omega_R} \{\, s \in \mathbb{Z} \mid \frac{-a}{c} \leq s \leq \frac{-b-r}{e} \,\} = \emptyset,$$

so the resolution is good for any Δ. Now, by Corollary 2.1.14 we have $H^q_{\mathcal{M}_\Delta}(R_\Delta) \cong H^{q+1}_{\mathcal{M}}(R)_\Delta = 0$ for $q < \dim R - 1$, so we are done. $\square$

We finish this section by giving necessary and sufficient conditions on the local cohomology of $\mathcal{R}_1$ and $\mathcal{R}_2$ for the existence of a Cohen-Macaulay diagonal of a standard bigraded Cohen-Macaulay k-algebra R. Namely,

Proposition 2.2.7 *Let R be a standard bigraded Cohen-Macaulay k-algebra of relevant dimension δ. Then there exists Δ such that R_Δ is Cohen-Macaulay if and only if $H^q_{\mathfrak{m}_1}(\mathcal{R}_1)_0 = H^q_{\mathfrak{m}_2}(\mathcal{R}_2)_0 = 0$ for any $q < \delta - 1$.*

Proof. According to Lemma 2.1.1, we have that $\dim R_\Delta = \delta - 1$ for any Δ. By taking into account Corollary 2.1.12, there exists Δ such that R_Δ is

Cohen-Macaulay if and only if there exists Δ such that $H^q_{\mathcal{M}_\Delta}(R_\Delta)_0 = 0$ for any $q < \delta - 1$. But from Proposition 2.2.1, for any $q < \delta - 1$ we have

$$H^q_{\mathcal{M}_\Delta}(R_\Delta)_0 \cong H^q_{\mathfrak{m}_1}(\mathcal{R}_1)_0 \oplus H^q_{\mathfrak{m}_2}(\mathcal{R}_2)_0.$$

This finishes the proof. □

2.3 Case study: Rees algebras

Let A be a noetherian graded algebra generated in degree 1 over a field k. Then A has a presentation $A = k[X_1, \ldots, X_n]/K = k[x_1, \ldots, x_n]$, where K is a homogeneous ideal of the polynomial ring $k[X_1, \ldots, X_n]$ with the usual grading. Let $\mathfrak{m}$ be the graded maximal ideal of A. For a homogeneous ideal I of A, let us consider the Rees algebra

$$R = R_A(I) = \bigoplus_{n \geq 0} I^n t^n \subset A[t]$$

of I endowed with the natural bigrading given by

$$R_{(i,j)} = (I^j)_i,$$

introduced by A. Simis et al. in [STV]. If I is generated by forms $f_1, \ldots, f_r$ in degrees $d_1, \ldots, d_r$ respectively, note that R is a k-algebra finitely generated by $x_1, \ldots, x_n, f_1t, ..., f_rt$ with $\deg(x_i) = (1, 0)$, $\deg(f_jt) = (d_j, 1)$, and that it has a unique homogeneous maximal ideal $\mathcal{M} = (x_1, \ldots, x_n, f_1t, ..., f_rt)$. By considering the polynomial ring $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ with the grading determined by setting $\deg(X_i) = (1, 0)$ and $\deg(Y_j) = (d_j, 1)$, we have a bigraded epimorphism:

$$\begin{array}{ccc} S & \longrightarrow & R \\ X_i & \longmapsto & x_i \\ Y_j & \longmapsto & f_jt \end{array}$$

so that R has a natural structure as finitely generated bigraded S-module. Set $d = \max\{d_1, \ldots, d_r\}$. For any $c \geq de + 1$, the $\Delta = (c, e)$-diagonal of the Rees algebra is

$$R_A(I)_\Delta = \bigoplus_{s \geq 0} (I^{es})_{cs} = k[(I^e)_c].$$

Note that $k[(I^e)_c]$ is a graded k-algebra with a unique homogeneous maximal ideal $m = \mathcal{M}_\Delta$. The interest of these algebras $k[(I^e)_c]$ is, as we will show in

the next chapter, that for any $c \geq de+1$ there is a projective embedding of the blow-up X of Proj (A) along $\widetilde{I}$ so that $X \cong$ Proj $(k[(I^e)_c])$.

Set $\overline{n} = \dim A$. The next lemma computes the dimension of the rings $k[(I^e)_c]$ extending [CHTV, Lemma 1.3] where the case $A = k[X_1, \dots, X_n]$ was studied.

Lemma 2.3.1 *Assume $I \not\subset \mathfrak{p}$, for all $\mathfrak{p} \in \mathrm{Ass}\,(A)$. Then $\dim k[(I^e)_c] = \overline{n}$ for all $c \geq de+1$.*

Proof. Since I is not contained in any associated prime of A, we have that any associated prime ideal of the Rees algebra R is relevant. So rel.dim $R = \dim R$. Furthermore, $\dim R = \overline{n}+1$ by [BH1, Exercise 4.4.12]. So we may conclude $\dim k[(I^e)_c] = \overline{n}$ by Lemma 2.1.1. □

From now on we will always assume that $I \not\subset \mathfrak{p}$, for all $\mathfrak{p} \in \mathrm{Ass}\,(A)$. The following result relates the local cohomology of the graded k-algebras $k[(I^e)_c]$ to the local cohomology of the Rees algebra. By setting $\mathfrak{n} = (It) = (f_1t, \dots, f_rt) \subset k[It] = k[f_1t, \dots, f_rt]$, we have

Proposition 2.3.2 *Let I be an ideal of A generated by forms of degree $\leq d$. For any diagonal $\Delta = (c, e)$ with $c \geq de+1$, there is a natural graded exact sequence*

$$\dots \to H^q_{\mathcal{M}}(R)_\Delta \to H^q_{\mathfrak{m}R}(R)_\Delta \oplus H^q_{\mathfrak{n}R}(R)_\Delta \to H^q_m(k[(I^e)_c]) \xrightarrow{\varphi^q_R} H^{q+1}_{\mathcal{M}}(R)_\Delta \to \dots$$

Proof. It is clear that $H^q_{\mathcal{M}_1}(R) = H^q_{\mathfrak{m}R}(R)$ and $H^q_{\mathcal{M}_2}(R) = H^q_{\mathfrak{n}R}(R)$. Then the result follows immediately by applying Proposition 2.1.3 to the Rees algebra R of I.□

In Corollary 2.1.4 we proved that the maps φ^q_L become isomorphisms for $q > \max\{n, r\}$. This bound was refined for standard bigraded k-algebras in Corollary 2.2.2. Next we want to consider the case of the Rees algebras. To this end, we are going to study the vanishing of the local cohomology modules of R with respect to $\mathfrak{n}R$. For any ideal I of A, the fiber cone of I is defined as the graded k-algebra $F = F_{\mathfrak{m}}(I) = \bigoplus_{n \geq 0} I^n/\mathfrak{m}I^n$. The analytic spread $l(I)$ of I is then the dimension of the fiber cone, that is, $l(I) = \dim F$. Note that if I is generated by forms of the same degree d, the fiber cone is nothing but $F_{\mathfrak{m}}(I) = k[I_d]$. The following lemma shows the known result that the local cohomology modules of R with respect to $\mathfrak{n}R$ vanish in order $q > l(I)$, but not in order $l(I)$. We include the proof for the sake of completeness.

Lemma 2.3.3 *Let I be a homogeneous ideal of A. Set $l = l(I)$. Then $H^q_{\mathfrak{n}R}(R) = 0$,$\forall q > l$ and $H^l_{\mathfrak{n}R}(R) \neq 0$.*

Proof. We may assume that the field k is infinite. Then there exists an ideal $J \subset I$ generated by $l = l(I)$ elements of A such that $I^m = JI^{m-1}$ for $m \gg 0$, that is, there exists a reduction J of I generated by l elements (see [BH1, Proposition 4.6.8]). Note that $(It)R$ and $(Jt)R$ are ideals with the same radical, so $H^q_{\mathfrak{n}R}(R) = H^q_{IR}(R) = H^q_{JR}(R) = 0, \forall q > l$. Moreover, from the presentation $R \to R/\mathfrak{m}R = F_{\mathfrak{m}}(I)$ we get the epimorphism

$$H^l_{\mathfrak{n}R}(R) \to H^l_{\mathfrak{n}R}(F) \neq 0,$$

so $H^l_{\mathfrak{n}R}(R) \neq 0$. $\square$

As a consequence, we get:

Corollary 2.3.4 *Let I be an ideal of A generated by forms in degree $\leq d$. For any Δ, we have a graded epimorphism*

$$H^{\overline{n}}_{m}(k[(I^e)_c]) \xrightarrow{\varphi^{\overline{n}}_R} H^{\overline{n}+1}_{\mathcal{M}}(R)_\Delta.$$

From Proposition 2.3.2 we may also deduce that for diagonals large enough the positive components of the local cohomology of the diagonals of the Rees algebra coincide with the positive components of the local cohomology of the powers of the ideal. Namely,

Corollary 2.3.5 *Let I be an ideal generated by forms of degree $\leq d$. For any $c \geq de + 1$, $e > a^2_*(R)$, $s > 0$, there are isomorphisms*

$$H^q_m(k[(I^e)_c])_s \cong H^q_{\mathfrak{m}}(I^{es})_{cs} \ \ ,\forall q \geq 0.$$

Proof. Let c, e be integers such that $c \geq de + 1$, $e > a^2_*(R)$. For any $s > 0$, we have that $H^q_{\mathcal{M}_1}(R)_{(cs,es)} = H^q_{\mathfrak{m}}(I^{es})_{cs}$ and $H^q_{\mathcal{M}_2}(R)_{(cs,es)} = 0$ by Proposition 2.1.18. Thus from Proposition 2.3.2 we get the isomorphisms $H^q_m(k[(I^e)_c])_s \cong H^q_{\mathfrak{m}}(I^{es})_{cs}$.$\square$

In the case where I is generated by forms in the same degree d (that is, I is equigenerated) the Rees algebra is a standard bigraded k-algebra by setting

$$R_A(I)_{(i,j)} = (I^j)_{i+dj}.$$

Then we may apply the results in Section 2.2 to these Rees algebras. From Lemma 2.2.4, we get a useful characterization for the Rees algebra to have a good resolution by means of the a_*-invariants of the ring A and the fiber cone of I. Namely,

Proposition 2.3.6 *Let I be an ideal of A generated by forms in degree d. The following are equivalent:*

(i) The Rees algebra $R_A(I)$ has a good resolution.

(ii) $a_*(A) < 0$, $a_*(F_{\mathfrak{m}}(I)) < 0$.

Proof. We have already noted that the Rees algebra is a standard bigraded ring by means of $R_{(i,j)} = (I^j)_{i+dj}$. With this grading, notice that $\mathcal{R}_1 = A$ and $\mathcal{R}_2 = k[I_d] = F_{\mathfrak{m}}(I)$. Then the result follows from Lemma 2.2.4. $\square$

To apply Proposition 2.3.6, we need to know the a_*-invariant of the fiber cone. The next two lemmas bound it. The first one gives a lower bound by means of the reduction number of I (compare with [Tr1], [Sch2]), while the second one gives an upper bound by means of the a_*-invariant of the Rees algebra.

Lemma 2.3.7 *Let $(A, \mathfrak{m})$ be a local noetherian ring with an infinite residue field. Let $I \subset \mathfrak{m}$ be an arbitrary ideal of A, J a minimal reduction of I and l the analytic spread of I. Then*

$$a_l(F_{\mathfrak{m}}(I)) + l \leq r_J(I) \leq \max\{a_i(F_{\mathfrak{m}}(I)) + i\} = \operatorname{reg}(F_{\mathfrak{m}}(I)).$$

Proof. Let $a_1, ..., a_l$ be a minimal system of generators of J. For $a \in I$, denote by a^0 the class of a in $I/\mathfrak{m}I$. Then $a_1^0, ..., a_l^0$ are a (homogeneous) system of parameters of $F_{\mathfrak{m}}(I)$ (see [HIO, Proposition 10.17]). According to [HIO, Lemma 45.1], we have

$$a_l(F_{\mathfrak{m}}(I)) + l \leq \max\{n \mid \left[\frac{F_{\mathfrak{m}}(I)}{(a_1^0, ..., a_l^0)}\right]_n \neq 0\} \leq \max\{a_i(F_{\mathfrak{m}}(I)) + i\}$$

On the other side,

$$\begin{aligned} r_J(I) &= \min\{n \mid I^{n+1} = JI^n\} \\ &= \min\{\ n \mid \tfrac{I^{n+1}}{\mathfrak{m}I^{n+1}} = \tfrac{JI^n + \mathfrak{m}I^{n+1}}{\mathfrak{m}I^{n+1}}\ \} \\ &= \min\{\ n \mid \left[\tfrac{F_{\mathfrak{m}}(I)}{(a_1^0, ..., a_l^0)}\right]_{n+1} = 0\ \} \end{aligned}$$

$$= \max\{ \ n \mid \left[\frac{F_{\mathfrak{m}}(I)}{(a_1^0,\ldots,a_l^0)}\right]_n \neq 0 \ \}.$$

This concludes the lemma. $\square$

The next lemma bounds the a_*-invariant of the fiber cone by means of the a_*-invariant of the Rees algebra. Namely,

Lemma 2.3.8 *Let I be an equigenerated homogeneous ideal of A. Then $a_*(F_{\mathfrak{m}}(I)) \leq a_*^2(R_A(I))$.*

Proof. Let

$$\mathbf{D}_. : \ 0 \to D_t \to \cdots \to D_1 \to D_0 = S \to R = R_A(I) \to 0$$

be the bigraded minimal free resolution of the Rees algebra R over S, where $D_p = \bigoplus_{(a,b)\in\Omega_p} S(a,b)$. Note that a, b are non-positive integers with $a \leq db$. Therefore, we have that

$$S(a,b)_0 = \bigoplus_j S_{(a+dj,b+j)} = \begin{cases} 0 & \text{if } a < db \\ S_2(b) & \text{if } a = db \end{cases}$$

Then by applying the functor $(\ )_0$ to the resolution $\mathbf{D}_.$ we get a graded free resolution of $R_0 = F_{\mathfrak{m}}(I)$ over S_2:

$$\mathbf{F}_. : \ 0 \to F_t \to \cdots \to F_1 \to F_0 = S_2 \to F \to 0 \ ,$$

where $F_p = (D_p)_0 = \bigoplus_{b\in\gamma_p} S_2(b)$, and $\gamma_p = \{b \in \mathbb{Z} : (db, b) \in \Omega_p\}$. Moreover, for any $p = 1, \ldots, t$ we have that $\mathrm{Im}(F_p) \subset \mathfrak{m}_2 F_{p-1}$, so $\mathbf{F}_.$ is in fact the graded minimal free resolution of $F_{\mathfrak{m}}(I)$ over S_2. Then by Theorem 1.3.4 we have:

$$\begin{aligned} a_*(F) &= \max\{-b \mid b \in \cup_p \gamma_p\} + a(S_2) \\ &\leq \max\{-b \mid (a,b) \in \Omega_R\} + a(S_2) \\ &= a_*^2(R). \quad \square \end{aligned}$$

Now we are ready to exhibit some families of ideals such that the diagonal functor and the local cohomology functor commute whenever we take diagonals large enough.

Example 2.3.9 Let I be an equigenerated ideal in a ring A with $a_*(A) < 0$ (for instance, we may take $A = k[X_1, \ldots, X_n]$). Set $r(I)$ the reduction number of I and assume that $F_{\mathfrak{m}}(I)$ is Cohen-Macaulay with negative a-invariant. Note that $a(F) < 0$ is equivalent to $r(I) < l(I)$ by Lemma 2.3.7. This class of ideals includes:

(i) ideals I with reduction number $r(I) = 0$ (for instance, complete intersection ideals and ideals of linear type).

(ii) $\mathfrak{m}$-primary ideals with $r(I) \le 1 < l(I)$ in a Cohen-Macaulay ring A (see [HS]) .

(iii) equimultiple ideals with $r(I) \le 1 < l(I)$ in a Cohen-Macaulay ring A (see [Sha]).

(iv) generically complete intersection ideals with $\mathrm{ad}(I) = 1$, $r(I) \le 1 < l(I)$ in a Cohen-Macaulay ring A (see [CZ]).

For these families of ideals, we have that the Rees algebra has a good resolution according to Lemma 2.3.6. Then we have graded isomorphisms

$$H^q_m(k[(I^e)_c]) \cong H^{q+1}_{\mathcal{M}}(R)_\Delta,$$

for any $q \ge 0$ and $c \gg e \gg 0$ by Corollary 2.1.14. Therefore, we have that for large diagonals $\mathrm{depth}(k[(I^e)_c]) \ge \mathrm{depth}(R) - 1$. In particular, if the Rees algebra is Cohen-Macaulay then its large diagonals will be also Cohen-Macaulay.

Remark 2.3.10 Recall that the form ring $G_A(I)$ of an ideal I in A is

$$G = G_A(I) = \bigoplus_{n \ge 0} I^n/I^{n+1} = R_A(I)/IR_A(I).$$

If I is a homogeneous ideal, the form ring has a natural bigrading by means of $G_{(i,j)} = (I^j/I^{j+1})_i$. We can get for the form ring similar results to the ones obtained for the Rees algebra. For instance, for an equigenerated ideal I we have that $G_A(I)$ has a good resolution if and only if $a_*(A/I) < 0$, $a_*(F_\mathfrak{m}(I)) < 0$ and it holds $a_*(F_\mathfrak{m}(I)) \le a^2_*(G_A(I))$.

Remark 2.3.11 For an equigenerated ideal I of A, note that we can recover several relationships between $r(I)$, $l(I)$ and $a^2(G)$ proved with more generality in [AHT]. By applying the diagonal functor to the minimal bigraded free resolution of $R_A(I)$ or $G_A(I)$, we obtain the minimal graded free resolution of $F_\mathfrak{m}(I) = k[I_d]$, and so $a_*(F_\mathfrak{m}(I)) \le a^2_*(R_A(I))$ and $a_*(F_\mathfrak{m}(I)) \le a^2_*(G_A(I))$. Now, according to Lemma 2.3.7, given J an arbitrary minimal reduction of I we have $r_J(I) - l(I) \le a^2_*(R_A(I))$ and the same formula for $G_A(I)$. In particular, if $R_A(I)$ is CM we get $r_J(I) \le l(I) + a^2(R_A(I)) \le l(I) - 1$. We can also obtain that if $R_A(I)$ is CM then $\mathrm{reltype}(I) \le \mu(I) - 1$.

Once we have studied the equigenerated case, in the rest of the section we consider the Rees algebra of an arbitrary homogeneous ideal I of A. In the case where A is the polynomial ring it was conjectured in [CHTV] that if the Rees algebra is Cohen-Macaulay there exist large diagonals which are Cohen-Macaulay. Note that for equigenerated ideals this follows from Proposition 2.3.6 and Lemma 2.3.8. Next we give a full answer to this conjecture.

Theorem 2.3.12 *Let I be a homogeneous ideal of the polynomial ring $A = k[X_1, \ldots, X_n]$. If $R_A(I)$ is Cohen-Macaulay, then $R_A(I)$ has a good resolution. In particular, $k[(I^e)_c]$ is Cohen-Macaulay for $c \gg e \gg 0$.*

Proof. Let us consider the bigraded minimal free resolution of R over S:

$$0 \to D_{r-1} \to \ldots \to D_1 \to D_0 = S \to R \to 0.$$

The first morphism in this resolution maps each X_i to X_i, so we immediately get that any shift $(a, b) \in \Omega_1$ satisfies $b < 0$. Then, by Lemma 1.3.1 jointly with Remark 1.3.3, we get that for any $p \geq 1$ and any shift $(a, b) \in \Omega_p$ we have $b < 0$. On the other hand, since R is Cohen-Macaulay, $a^2_*(R) = a^2(R)$. Now, by Lemma 1.2.3 we have $a^2(R) = a(R_2)$, where $R_2 = \bigoplus_j (\bigoplus_i (I^j)_i) = \bigoplus_j I^j$ is nothing but the Rees algebra with the usual $\mathbb{Z}$- grading, and so $a(R_2) = -1$. By then applying Theorem 1.3.4, for any $(a, b) \in \Omega_R$ we have

$$-b \leq a^2(S) - a^2_*(R) = r - 1 < r,$$

so by Proposition 2.1.16 the Rees algebra has a good resolution. Then the result follows from Corollary 2.1.14. □

Assume that a Rees algebra is Cohen-Macaulay. Is there any Cohen-Macaulay diagonal? We know that this holds if A is the polynomial ring. The next result provides the obstruction for the existence of such diagonals in the general case. Note that the obstruction depends only on the ring A.

Theorem 2.3.13 *If $R_A(I)$ is Cohen-Macaulay, then the following are equivalent:*

(i) There exist c, e such that $k[(I^e)_c]$ is Cohen-Macaulay.

(ii) $H^i_{\mathfrak{m}}(A)_0 = 0$ for all $i < \overline{n}$.

Proof. If R is Cohen-Macaulay, then $a^2_*(R) = a^2(R) = -1$, so by Proposition 2.1.18 we have $H^q_{\mathfrak{n}R}(R)_{(0,0)} = 0$ for all q. Then, according to Proposition 2.3.2 and Proposition 2.1.18, $H^q_m(k[(I^e)_c])_0 = H^q_{\mathfrak{m}R}(R)_{(0,0)} = H^q_{\mathfrak{m}}(A)_0$. Now the statement follows from Corollary 2.1.12. □

Chapter 3

Cohen-Macaulay coordinate rings of blow-up schemes

After introducing in the previous chapters the basic tools we will need along this work, we are now ready to study in detail the Cohen-Macaulayness of the coordinate rings of blow-ups of projective varieties. Let k be a field, and let Y be a closed subscheme of $\mathbb{P}_k^{n-1}$ with coordinate ring $A = k[X_1, \ldots, X_n]/K$, where $K \subset k[X_1, \ldots, X_n]$ is a homogeneous ideal. Given $I \subset A$ a homogeneous ideal, let denote by $\mathcal{I} = \widetilde{I}$ the sheaf associated to I in $Y = \text{Proj}\ (A)$. Let X be the projective scheme obtained by blowing up Y along $\mathcal{I}$, that is, $X = \mathcal{P}roj(\bigoplus_{n\geq 0} \mathcal{I}^n)$. If I is generated by forms of degree $\leq d$, then $(I^e)_c$ corresponds to a complete linear system on X very ample for $c \geq de+1$ which gives a projective embedding of X so that $X \cong \text{Proj}\ (k[(I^e)_c]) \subset \mathbb{P}_k^{N-1}$, where $N = \dim_k (I^e)_c$ (see [CH, Lemma 1.1]).

For a given homogeneous ideal $I \subset A$, we can consider the Rees algebra $R_A(I) = \oplus_{j\geq 0} I^j$ of I endowed with the natural bigrading $R_A(I)_{(i,j)} = (I^j)_i$. By taking diagonals $\Delta = (c, e)$ with $c \geq de+1$, we have that $R_A(I)_\Delta = k[(I^e)_c]$. In Chapter 2 we used this fact to study the existence of algebras $k[(I^e)_c]$ which are Cohen-Macaulay in the case where the Rees algebra also has this property (see Theorems 2.3.12 and 2.3.13). Our aim in this chapter is to get some general criteria for the existence of (at least) one coordinate ring $k[(I^e)_c]$ with the Cohen-Macaulay property. In Section 3.2 we will give sufficient and necessary conditions to ensure this existence by means of the local cohomology of $R_A(I)$ and the sheaf cohomology $H^i(\mathrm{X}, \mathcal{O}_\mathrm{X})$. This result will be applied in

Section 3.3 to exhibit several situations where we can ensure the existence of Cohen-Macaulay coordinate rings for X. We also give a criterion for the existence of Buchsbaum coordinate rings, proving in particular a conjecture stated by A. Conca et al. [CHTV].

Once we have studied the existence of Cohen-Macaulay diagonals of a Rees algebra, in Section 3.4 our aim will be to precise these diagonals. This is a difficult problem which has only been completely solved for complete intersection ideals in the polynomial ring [CHTV, Theorem 4.6]. We will give several criteria to decide if a given diagonal is Cohen-Macaulay, which will allow us to recover and extend the result on complete intersection ideals as well as to determine the Cohen-Macaulay diagonals for new families of ideals. If the Rees algebra is Cohen-Macaulay, we can also determine a family of Cohen-Macaulay diagonals. The section finishes by studying the coordinate rings of the embeddings of the blow-up of a projective space along an ideal of fat points.

The last section is devoted to study sufficient conditions for the existence of a constant f ensuring that $k[(I^e)_c]$ is Cohen-Macaulay for any $c \geq ef$ and $e > 0$, a question that has been treated by S.D. Cutkosky and J. Herzog in [CH]. The main result shows that this holds for homogeneous ideals in a Cohen-Macaulay ring A whose Rees algebra is Cohen-Macaulay at any $\mathfrak{p} \in$ Proj (A).

3.1 The blow-up of a projective variety

From now on in this chapter we will have the following assumptions. Let k be a field and A a noetherian graded k-algebra generated in degree 1. Then A has a presentation $A = k[X_1, \dots, X_n]/K = k[x_1, \dots, x_n]$, where K is a homogeneous ideal in the polynomial ring $k[X_1, \dots, X_n]$ with the usual grading. We will denote by $\mathfrak{m}$ the graded maximal ideal of A. Let Y be the projective scheme $\mathrm{Proj}(A) \subset \mathbb{P}_k^{n-1}$. Let I be a homogeneous ideal not contained in any associated prime ideal of A, and let $\mathcal{I}$ be the sheaf associated to I in Y. Then $\mathcal{I}$ can be blown up to produce the projective scheme $X = \mathcal{P}roj(\bigoplus_{n \geq 0} \mathcal{I}^n)$ together with a natural morphism $\pi : X \to Y$. Let us recall the construction of the $\mathcal{P}roj$ of a sheaf of graded algebras $\mathcal{R}$ over a scheme Y (see [Har, Chapter II, Section 7]). For each open affine subset $U = \mathrm{Spec}(B)$ of Y, let $\mathcal{R}(U)$ be

the graded B-algebra $\Gamma(U, \mathcal{R}|U)$. Then we can consider $\mathrm{Proj}(\mathcal{R}(U))$ and its natural morphism $\pi|U : \mathrm{Proj}(\mathcal{R}(U)) \to U$. These schemes can be glued to obtain the scheme $\mathcal{P}roj(\mathcal{R})$ with the morphism $\pi : \mathcal{P}roj(\mathcal{R}) \to Y$ such that for each open affine $U \subset Y$, $\pi^{-1}(U) \cong \mathrm{Proj}(\mathcal{R}(U))$.

Assume that I is generated by forms $f_1, \ldots, f_r$ in degrees $d_1, \ldots, d_r$ respectively. Let $d = \max\{d_1, \ldots, d_r\}$. For any $c \geq d+1$, let us consider the invertible sheaf of ideals $\mathcal{L} = \mathcal{I}(c)\mathcal{O}_X$. We are going to show that $\mathcal{L}$ defines a morphism of X in a projective space $\varphi : X \to \mathbb{P}_k^{N-1}$ which is a closed immersion so that $X \cong \mathrm{Proj}\ (k[I_c])$. Since the blow-up of Y along $\mathcal{I}^e$ is isomorphic to X, we will also have $X \cong \mathrm{Proj}\ (k[(I^e)_c])$ for any $c \geq de+1$. For that, we are going to follow the proof of [CH, Lemma 1.1]. First of all, notice that we have an affine cover of X by considering the set $\{U_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq r\}$, where $U_{ij} = \mathrm{Spec}(R_{ij})$, and

$$R_{ij} = \left(k\left[\frac{X_1}{X_i}, \ldots, \frac{X_n}{X_i}\right]/K_i\right)\left[\frac{f_1 X_i^{d_j - d_1}}{f_j}, \ldots, \frac{f_r X_i^{d_j - d_r}}{f_j}\right].$$

Furthermore, $\Gamma(U_{ij}, \mathcal{I}(c)\,\mathcal{O}_X) = f_j x_i^{c-d_j} R_{ij}$. Since $f_j x_i^{c-d_j} \in I_c$ and $I_c \subset \Gamma(X, \mathcal{I}(c)\,\mathcal{O}_X)$, we have that $\mathcal{L} = (I_c)\,\mathcal{O}_X$.

I_c is a k-vector space generated by the elements s of the type $s = f_j x_1^{l_1} \ldots x_n^{l_n}$ with degree c, that is, such that $d_j + l_1 + \ldots + l_n = c$. By considering $X_s = \{P \in X \mid s_P \notin \mathfrak{m}_P \mathcal{L}_P\}$ with $s \in I_c$, we have an open covering of X. Since $c > d$, there exists some i with $l_i > 0$, so denoting by $u = (\frac{x_1}{x_i})^{l_1} \ldots (\frac{x_n}{x_i})^{l_n}$ we have that

$$X_s = \mathrm{Spec}((R_{ij})_u) = \mathrm{Spec}\left(R_{ij}\left[\left(\frac{x_i}{x_1}\right)^{l_1} \ldots \left(\frac{x_i}{x_n}\right)^{l_n}\right]\right)$$

is an open affine.

Set $N = \dim_k I_c$. Let $\mathbb{P}_k^{N-1} = \mathrm{Proj}\ (k[\{Z_s\}_{s \in \Lambda}])$, where Λ is a k-basis of I_c, and $V_s = D_+(Z_s) \subset \mathbb{P}_k^{N-1}$. The k-linear maps defined by

$$\begin{array}{ccc} \Gamma(V_s, \mathcal{O}_{V_s}) = k[T_{\overline{s}} : \overline{s} \neq s] & \longrightarrow & \Gamma(X_s, \mathcal{O}_{X_s}) \\ T_{\overline{s}} & \mapsto & \frac{\overline{s}}{s} \end{array}$$

are epimorphisms which define morphisms of schemes $X_s \to V_s$. By gluing them, we get a closed immersion $\varphi : X \to \mathbb{P}_k^{N-1}$ so that $X \cong \mathrm{Proj}\ (k[I_c])$ by [Har, Proposition II.7.2].

Let $\mathcal{L} = \widetilde{I}\mathcal{O}_X$, $\mathcal{M} = \pi^*\mathcal{O}_Y(1)$, so that $(I^e)_c\mathcal{O}_X = \mathcal{L}^e \otimes \mathcal{M}^c$. Then, the classical Serre's exact sequence allows to relate the local cohomology of the rings $k[(I^e)_c]$ and the global cohomology.

Remark 3.1.1 *[CH, Lemma 1.2]* There is an exact graded sequence

$$0 \to H^0_m(k[(I^e)_c]) \to k[(I^e)_c] \to \bigoplus_{s\in\mathbb{Z}} \varGamma(\mathrm{X}, \mathcal{L}^{es} \otimes \mathcal{M}^{cs}) \to H^1_m(k[(I^e)_c]) \to 0$$

and isomorphisms

$$H^i_m(k[(I^e)_c]) \cong \bigoplus_{s\in\mathbb{Z}} H^{i-1}(\mathrm{X}, \mathcal{L}^{es} \otimes \mathcal{M}^{cs})$$

for $i > 1$. In particular, looking at the homogeneous component of degree 0 we get the exact sequence

$$0 \to H^0_m(k[(I^e)_c])_0 \to k \to \varGamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) \to H^1_m(k[(I^e)_c])_0 \to 0$$

and isomorphisms $H^i_m(k[(I^e)_c])_0 \cong H^{i-1}(\mathrm{X}, \mathcal{O}_{\mathrm{X}})$ for $i > 1$.

For a homogenous ideal I of A, let us consider the Rees algebra $R = R_A(I) = \bigoplus_{n\geq 0} I^n t^n \subset A[t]$ of I endowed with the natural bigrading given by $R_{(i,j)} = (I^j)_i$. Then, by taking a diagonal $\Delta = (c, e)$ with $c \geq de + 1$, we have that $R_A(I)_\Delta = \bigoplus_{s\geq 0}(I^{es})_{cs} = k[(I^e)_c]$. The natural inclusion $k[(I^e)_c] = R_\Delta \hookrightarrow R$, gives the isomorphism of schemes $\mathrm{Proj}\ ^2(R_A(I)) \cong \mathrm{Proj}\ (k[(I^e)_c])$. Summarizing, we have:

Proposition 3.1.2 *Let X be the blow-up of $Y = \mathrm{Proj}\ (A)$ along $\mathcal{I} = \widetilde{I}$, where I is a homogeneous ideal of A generated by forms of degree $\leq d$. For any $c \geq de + 1$, we have isomorphisms of schemes*

$$X \cong \mathrm{Proj}\ ^2(R_A(I)) \cong \mathrm{Proj}\ (k[(I^e)_c]).$$

In Chapter 2, Section 3, we have followed an algebraic approach to study the local cohomology modules of the rings $k[(I^e)_c]$ in terms of the local cohomology of the Rees algebra and the diagonal functor. Next we give a new approach to these modules by using sheaf cohomology.

Notice that from Remark 3.1.1 we may determine the local cohomology modules of the k-algebras $k[(I^e)_c]$ by means of the cohomology modules

$H^i(\mathrm{X}, \mathcal{L}^{es} \otimes \mathcal{M}^{cs})$. On the other hand, we can get some information about these modules from the Leray spectral sequence

$$E_2^{i,j} = H^i(\mathrm{Y}, R^j\pi_*(\mathcal{L}^{es} \otimes \mathcal{M}^{cs})) \Rightarrow H^{i+j}(\mathrm{X}, \mathcal{L}^{es} \otimes \mathcal{M}^{cs}),$$

and the vanishing of the higher direct-image sheaves $R^j\pi_*(\mathcal{L}^{es} \otimes \mathcal{M}^{cs})$. First of all, let us study the vanishing of $R^j\pi_*(\mathcal{L}^{es})$.

Theorem 3.1.3 *Set* $e_0 = \max\{\, a_*(R_{A_\mathfrak{p}}(I_\mathfrak{p})) : \mathfrak{p} \in \mathrm{Proj}\ (A)\,\}$. *For any* $e > e_0$, $j > 0$, $R^j\pi_*\mathcal{L}^e = 0$ *and* $\pi_*\mathcal{L}^e = \widetilde{I^e}$.

Proof. Let us denote by $A_i = A_{(x_i)}$, $I_i = I_{(x_i)}$ and $R_i = R_{A_i}(I_i) = A_i[I_it]$. Note that by defining $Y_i = Y - V_+(x_i) = D_+(x_i) \cong \mathrm{Spec}(A_i)$, we have that $\{Y_i : 1 \le i \le n\}$ is an open affine cover of Y. Then, given $j > 0, e > 0$, $R^j\pi_*\mathcal{L}^e = 0$ if and only if $(R^j\pi_*\mathcal{L}^e) \mid Y_i = 0$ for all i. Denoting by $X_i = \pi^{-1}Y_i = \mathrm{Proj}\ (R_i)$, by [Har, Corollary III.8.2 and Proposition III.8.5] we have that for $j > 0$

$$R^j\pi_*(\mathcal{L}^e) \mid Y_i = R^j\pi_*(\mathcal{L}^e \mid X_i) = H^j(X_i, \mathcal{L}^e \mid \mathrm{X}_i)^{\sim} = H^{j+1}_{(R_i)_+}((I_i)^eR_i)_0{}^{\sim}\,.$$

From the graded exact sequence

$$0 \to (I_i)^eR_i(-e) \to R_i \to \bigoplus_{q<e}(I_i)^q \to 0\ ,$$

it follows that $H^{j+1}_{(R_i)_+}((I_i)^eR_i)_0 = H^{j+1}_{(R_i)_+}((I_i)^eR_i(-e))_e = H^{j+1}_{(R_i)_+}(R_i)_e$. Similarly, $\pi_*\mathcal{L}^e = \widetilde{I^e}$ if $H^0_{(R_i)_+}(R_i)_e = H^1_{(R_i)_+}(R_i)_e = 0$ for all i. Therefore, we have reduced the problem to prove that $H^j_{(R_i)_+}(R_i)_e = 0$ for all i, j if $e > e_0$.

Set $\overline{R}_i = R_{A_{x_i}}(I_{x_i})$. We can think $\overline{R}_i$ as a $\mathbb{Z}$-graded ring with $\deg(\frac{x_j}{x_i^m}) = 1 - m$, $\deg(\frac{f_jt}{x_i^m}) = d_j - m$. Note that with this grading we have $[\overline{R}_i]_0 = R_i$ and $\frac{x_i}{1}$ is an invertible element in $\overline{R}_i$ of degree 1. Then we may define the graded isomorphism

$$\begin{array}{ccc} R_i[T, T^{-1}] & \xrightarrow{\psi} & \overline{R}_i \\ T & \mapsto & \frac{x_i}{1}\ , \end{array}$$

where $\psi|R_i = id$ and $\deg(T) = 1$. Since $R_i \hookrightarrow \overline{R}_i$ is a flat morphism, we have that

$$H^j_{(\overline{R}_i)_+}(\overline{R}_i) = H^j_{(R_i)_+}(R_i) \otimes_{R_i} \overline{R}_i = H^j_{(R_i)_+}(R_i)[T, T^{-1}],$$

so that $H^j_{(\overline{R}_i)_+}(\overline{R}_i)_e = H^j_{(R_i)_+}(R_i)_e\,[T, T^{-1}]$. Therefore, it suffices to prove that $H^j_{(\overline{R}_i)_+}(\overline{R}_i)_e = 0$ for all i, j if $e > e_0$.

Given a homogeneous prime $\mathfrak{q} \in \operatorname{Spec}(A_{x_i})$, we have that $\mathfrak{q} = \mathfrak{p} A_{x_i}$ with $\mathfrak{p} \in \operatorname{Proj}(A)$. Localizing $\overline{R}_i$ at $\mathfrak{q}$, we have that $\overline{R}_i \otimes (A_{x_i})_{\mathfrak{q}} = R_{A_{\mathfrak{p}}}(I_{\mathfrak{p}})$. Denoting by $B = R_{A_{\mathfrak{p}}}(I_{\mathfrak{p}})$, note that B is a standard graded ring whose homogeneous component of degree 0 is the local ring $A_{\mathfrak{p}}$. So B has a unique homogeneous maximal ideal $\mathfrak{n}$, with $\mathfrak{n} = \mathfrak{p} A_{\mathfrak{p}} \oplus B_+$. Since $H^j_{\mathfrak{n}}(B)_e = 0$ for all $j \geq 0$ and $e > e_0$, according to [Hy, Lemma 2.3] we also have $H^j_{B_+}(B)_e = 0$ for all $j \geq 0$ and $e > e_0$. Therefore,

$$[H^j_{(\overline{R}_i)_+}(\overline{R}_i)_e]_{\mathfrak{q}} = [H^j_{(\overline{R}_i)_+}(\overline{R}_i)_{\mathfrak{q}}]_e = H^j_{B_+}(B)_e = 0.$$

Hence $(H^j_{(\overline{R}_i)_+}(\overline{R}_i)_e)_{\mathfrak{q}} = 0$ for any homogeneous ideal $\mathfrak{q} \in \operatorname{Spec}(A_{x_i})$, and we conclude $H^j_{(\overline{R}_i)_+}(\overline{R}_i)_e = 0$ for $j \geq 0$ and $e > e_0$. $\square$

Corollary 3.1.4 *Assume that $R_{A_{\mathfrak{p}}}(I_{\mathfrak{p}})$ is Cohen-Macaulay for any $\mathfrak{p} \in \operatorname{Proj}(A)$. Then, for any $e \geq 0$, $j > 0$, $R^j \pi_* \mathcal{L}^e = 0$ and $\pi_* \mathcal{L}^e = \widetilde{I^e}$.*

Given a homogeneous ideal I of A, let us denote by $I^* = \{f \in A \mid \mathfrak{m}^k f \subset I \text{ for some } k\}$ the saturation of I. Note that $H^0_{\mathfrak{m}}(A/I) = I^*/I$. Next we use Theorem 3.1.3 to relate the local cohomology modules of the k-algebras $k[(I^e)_c]$ and the local cohomology of the powers of the ideal (compare with Corollary 2.3.5).

Corollary 3.1.5 *Let I be a homogeneous ideal of A, and $e_0 = \max\{\, a_*(R_{A_{\mathfrak{p}}}(I_{\mathfrak{p}})) : \mathfrak{p} \in \operatorname{Proj}(A)\,\}$. For any $c \geq de+1$, $e > e_0$, $s > 0$, there is an exact sequence*

$$0 \to H^0_m(k[(I^e)_c])_s \to (I^{es})_{cs} \to (I^{es})^*_{cs} \to H^1_m(k[(I^e)_c])_s \to 0$$

and isomorphisms $H^i_m(k[(I^e)_c])_s \cong H^i_{\mathfrak{m}}(I^{es})_{cs}$ for $i > 1$.

Proof. By the Leray spectral sequence we have

$$H^i(\mathrm{Y}, R^j \pi_*(\mathcal{L}^{es} \otimes \mathcal{M}^{cs})) \Rightarrow H^{i+j}(\mathrm{X}, \mathcal{L}^{es} \otimes \mathcal{M}^{cs}).$$

On the other hand, by the Projection formula [Har, Exercise III.8.3] and Theorem 3.1.3, we get that for $e > e_0$, $s > 0$,

$$\pi_*(\mathcal{L}^{es} \otimes \mathcal{M}^{cs}) = \pi_*(\mathcal{L}^{es}) \otimes \mathcal{O}_{\mathrm{Y}}(cs) = \widetilde{I^{es}}(cs)$$

$$R^j \pi_*(\mathcal{L}^{es} \otimes \mathcal{M}^{cs}) = R^j \pi_*(\mathcal{L}^{es}) \otimes \mathcal{O}_{\mathrm{Y}}(cs) = 0, \text{ for all } j > 0.$$

Therefore we may conclude that for any $i \geq 1$, $H^i(\mathrm{X}, \mathcal{L}^{es} \otimes \mathcal{M}^{cs}) = H^i(\mathrm{Y}, \widetilde{I^{es}}(cs)) = H^{i+1}_{\mathfrak{m}}(I^{es})_{cs}$, and $\Gamma(X, \mathcal{L}^{es} \otimes \mathcal{M}^{cs}) = \Gamma(Y, \widetilde{I^{es}}(cs)) = (I^{es})^*_{cs}$. Now the result follows from Remark 3.1.1. □

We have shown that the positive components of the local cohomology modules of the rings $k[(I^e)_c]$ are closely related to the local cohomology modules of the powers of the ideal I^e. Next we want to study the negative components. In the case where X is Cohen-Macaulay, we will express them by means of the local cohomology of the canonical module of the Rees algebra. Recall that the canonical module of the Rees algebra is defined in the category of bigraded S-modules, so let us write $K_R = \bigoplus_{(i,j)} K_{(i,j)}$. For any integer e, we denote by $K^e = (K_R)^e = \bigoplus_i K_{(i,e)}$. Then we have

Proposition 3.1.6 *Assume that X is an equidimensional Cohen-Macaulay scheme. For any $c \geq de+1$, $e > a^2_*(K_R)$, $s > 0$, $1 \leq i < \overline{n}$,*

$$H^i_m(k[(I^e)_c])_{-s} \cong H^{\overline{n}-i+1}_{\mathfrak{m}}(K^{es})_{cs}.$$

Proof. By Serre's duality and Remark 3.1.1

$$H^i_m(k[(I^e)_c])_{-s} \cong H^{i-1}(X, \mathcal{L}^{-es} \otimes \mathcal{M}^{-cs}) \cong H^{\overline{n}-i}(X, w_X \otimes \mathcal{L}^{es} \otimes \mathcal{M}^{cs}).$$

Then, by taking $c \geq de+1$, $e > a^2_*(K_R)$, $i < \overline{n}$ we get

$$\begin{aligned} H^i_m(k[(I^e)_c])_{-s} &= H^{\overline{n}+1-i}_{R_+}(K_R)_{(cs,es)} \\ &= H^{\overline{n}+1-i}_{\mathfrak{m}}(K^{es})_{cs}, \text{ by Proposition 2.1.18. } \square \end{aligned}$$

Remark 3.1.7 According to [HHK, Theorem 2.1] we can also express the negative components of the local cohomology of the rings $k[(I^e)_c]$ by means of the local cohomology of their canonical modules whenever X is Cohen-Macaulay equidimensional. In this case, there are also isomorphisms $H^i_m(k[(I^e)_c])_{-s} \cong H^{\overline{n}+1-i}_m(K_{R_\Delta})_s$ for any $s > 0$ and $1 \leq i < \overline{n}$.

But notice that $[K_{R_\Delta}]_j \cong [(K_R)_\Delta]_j$ for any $j \gg 0$, so we immediately get

$$H^i_m(K_{R_\Delta})_s \cong H^i_m((K_R)_\Delta)_s \cong H^i_{R_+}(K_R)_{(cs,es)} \cong H^i_{\mathfrak{m}}(K^{es})_{cs}$$

for any $s \geq 0$, $i > 1$.

3.2 Existence of Cohen-Macaulay coordinate rings

Our aim in this section is to find necessary and sufficient conditions for the existence of integers c, e, with $c \geq de + 1$, such that the ring $k[(I^e)_c]$ is Cohen-Macaulay. Before proving our main result, we need two previous lemmas. The first one may be seen as a Nakayama's Lemma adapted to our situation, and in fact it is just Lemma 1.5.2 for the case $r = 2$.

Lemma 3.2.1 *Let L be a finitely generated bigraded R-module and m an integer such that $R_+^m L$ =0. Then, there exist integers q_0, t such that $L_{(p,q)}$=0 for all $p > dq + t$, $q > q_0$.*

The second lemma provides restrictions on the local cohomology modules of the Rees algebra whenever X is Cohen-Macaulay.

Lemma 3.2.2 *If X is Cohen-Macaulay equidimensional, then there are integers q_0, t such that $H^i_{\mathcal{M}}(R_A(I))_{(p,q)} = 0$ for all $i < \overline{n} + 1$, $q < q_0$ and $p < dq + t$.*

Proof. Let $\mathrm{P} \in X$. Then $R_+ \not\subset \mathrm{P}$ and so there exist i, j such that $x_i \notin \mathrm{P}$, $f_j t \notin \mathrm{P}$. Denote by $R_{<\mathrm{P}>} = T^{-1}R$, where T is the multiplicative system consisting of all homogeneous elements of R which are not in P. Note that $R_{(\mathrm{P})} = [R_{<\mathrm{P}>}]_{(0,0)}$. Furthermore, $\frac{x_i}{1}$ and $\frac{f_j t}{x_i^{d_j}}$ are invertible elements in $R_{<\mathrm{P}>}$ with

$$\deg \frac{x_i}{1} = (1,0), \ \deg \ \frac{f_j t}{x_i^{d_j}} = (0,1).$$

Then we may define a bigraded isomorphism ψ:

$$\begin{array}{ccc} R_{(\mathrm{P})}[U, U^{-1}, V, V^{-1}] & \xrightarrow{\psi} & R_{<\mathrm{P}>} \\ U & \longmapsto & \frac{x_i}{1} \\ V & \longmapsto & \frac{f_j t}{x_i^{d_j}} \end{array}$$

where $\psi | R_{(\mathrm{P})} = id$, and $\deg(U) = (1,0)$, $\deg(V) = (0,1)$. Since X is CM, $\mathcal{O}_{\mathrm{X,P}} = R_{(\mathrm{P})}$ is CM and so $R_{<\mathrm{P}>}$ too. Then, localizing at $\mathrm{P}R_{<\mathrm{P}>}$, we have that R_P is CM.

Now let $\mathrm{P} \in \mathrm{Spec}(R)$ and denote by P^* the ideal generated by the homogeneous elements of P. By [GW2, Corollary 1.2.4], R_P is CM if and only if R_{P^*} is CM, so we have that R_P is CM for any prime ideal P such that $R_+ \not\subset \mathrm{P}$.

Localizing the Rees algebra R at the homogeneous maximal ideal $\mathcal{M}$ we then have that $R_{\mathcal{M}}$ is a generalized Cohen-Macaulay module with respect to $R_+R_{\mathcal{M}}$ [HIO, Lemma 43.2]. Therefore there exists $m \geq 0$ such that $R_+^m H^i_{\mathcal{M}}(R) = 0$ for all $i < \overline{n}+1$. From the presentation of R as a quotient of the polynomial ring $S = k[X_1, \dots, X_n, Y_1, \dots, Y_r]$, by Theorem 1.2.1 we get

$$H^i_{\mathcal{M}}(R) = \underline{\mathrm{Ext}}_S^{n+r-i}(R, K_S)^{\vee},$$

and so $R_+^m \underline{\mathrm{Ext}}_S^{n+r-i}(R, K_S) = 0$ for $i < \overline{n}+1$. By Lemma 3.2.1 we then obtain that there exist integers q_1, t_1 such that $\underline{\mathrm{Ext}}_S^{n+r-i}(R, K_S)_{(p,q)} = 0$ for all $q > q_1$, $p > dq + t_1$ and $i < \overline{n}+1$. The proof finishes by dualizing again. $\square$

Now we may formulate the main result of this section.

Theorem 3.2.3 *The following are equivalent:*

(i) *There exist c, e such that $k[(I^e)_c]$ is Cohen-Macaulay.*

(ii) *(1) There exist integers q_0, t such that $H^i_{\mathcal{M}}(R_A(I))_{(p,q)} = 0$ for all $i < \overline{n}+1$, $q < q_0$ and $p < dq + t$.*

(2) $\Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) = k$ and $H^i(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) = 0$ for $0 < i < \overline{n}-1$.

In this case, $k[(I^e)_c]$ is Cohen-Macaulay for $c \gg 0$ relatively to $e \gg 0$.

Proof. If (i) is satisfied, then the scheme $X = \mathrm{Proj}\ ^2(R) \cong \mathrm{Proj}\ (k[(I^e)_c])$ is CM and equidimensional, so we have (1) of (ii) by Lemma 3.2.2. Furthermore, $H^i_m(k[(I^e)_c])_0 = 0$ for any $i < \overline{n}$ and then by using Remark 3.1.1 we get (2) of (ii).

Assume now that (ii) is satisfied. We want to find a diagonal Δ such that $R_\Delta = k[(I^e)_c]$ is CM. By Remark 3.1.1 and (2) of (ii), we have that $H^i_m(R_\Delta)_0 = 0$ for any diagonal Δ and $i < \overline{n}$. On the other hand, since $H^i_{\mathcal{M}}(R)$ are artinian modules there exists p_1 such that $H^i_{\mathcal{M}}(R)_{(p,q)} = 0$ for all i and $p > p_1$. Furthermore, by Corollary 2.1.12 there are positive integers e_0, α such that for $e > e_0$, $c > de + \alpha$ we have

$$H^i_m(R_\Delta)_j \cong H^{i+1}_{\mathcal{M}}(R)_{(cj,ej)}, \forall i, \forall j \neq 0.$$

Now, let us consider q_0, t given by (1) of (ii). Note that we can assume that q_0, t are negative. Then, by taking diagonals $\Delta = (c, e)$ with $e > \max\{e_0, -q_0\}$, $c > \max\{de + \alpha, p_1, de - t\}$, we have that $H^i_m(R_\Delta)_j = 0$ for all j and $i < \overline{n}$, and therefore $k[(I^e)_c]$ are CM for all these c, e. $\square$

Remark 3.2.4 Assume that $(A, \mathfrak{m})$ is a noetherian local ring and let $I \subset \mathfrak{m}$, $I \neq 0$ be an ideal. Denote by $X = \text{Proj}\,(R_A(I))$ the blow-up of $\text{Spec}(A)$ along I. Then, it was proved by J. Lipman [Li, Theorem 4.1] that there exists a positive integer e such that $R_A(I^e)$ is Cohen-Macaulay if and only if X is Cohen-Macaulay, $\Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) = A$ and $H^i(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) = 0$ for all $i > 0$. The following corollary may be seen as a projective version of this result.

Corollary 3.2.5 *The following are equivalent:*

(i) There exist c, e such that $k[(I^e)_c]$ is Cohen-Macaulay.

(ii) X is a Cohen-Macaulay equidimensional scheme, $\Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) = k$ and $H^i(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) = 0$ for all $0 < i < \overline{n} - 1$.

(iii) X is Cohen-Macaulay equidimensional and $H^i_{R_+}(R)_{(0,0)} = 0$ for $i < \overline{n}$.

Proof. It is enough to note that we have an exact bigraded sequence

$$0 \to H^0_{R_+}(R) \to R \to \bigoplus_{(p,q)} \Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}}(p,q)) \to H^1_{R_+}(R) \to 0,$$

and isomorphisms $H^{i+1}_{R_+}(R) \cong \bigoplus_{(p,q)} H^i(\mathrm{X}, \mathcal{O}_{\mathrm{X}}(p,q))$ for $i > 0$. $\square$

We can also give sufficient and necessary conditions for the existence of generalized Cohen-Macaulay or Buchsbaum diagonals of the Rees algebra, in particular proving a conjecture of A. Conca et al. in [CHTV].

Proposition 3.2.6 *The following are equivalent:*

(i) $H^i_{\mathcal{M}}(R_A(I))_{(p,q)} = 0$ for $i < \overline{n} + 1$ and $p \ll 0$ relatively to $q \ll 0$.

(ii) $k[(I^e)_c]$ is a generalized Cohen-Macaulay module for $c \gg e \gg 0$.

(iii) There exist c, e such that $k[(I^e)_c]$ is generalized Cohen-Macaulay.

(iv) $k[(I^e)_c]$ is a Buchsbaum ring for $c \gg e \gg 0$.

(v) There exist c, e such that $k[(I^e)_c]$ is a Buchsbaum ring.

(vi) There exist integers q_0, t such that $H^i_{\mathcal{M}}(R_A(I))_{(p,q)} = 0$ for $i < \overline{n} + 1$, $q < q_0$ and $p < dq + t$.

Proof. $(i) \Rightarrow (ii)$ Assume that (i) is satisfied. By Corollary 2.1.12, we get $H^i_m(R_\Delta)_s = 0$ for $c \gg 0$ relatively to $e \gg 0$, $s \neq 0$ and $i < \overline{n}$. So $k[(I^e)_c]$ is generalized CM for $c \gg e \gg 0$.

$(ii) \Rightarrow (iii)$ Obvious.

$(iii) \Rightarrow (vi)$ Let Δ be a diagonal such that R_Δ is generalized CM. Then $(R_\Delta)_\mathfrak{p}$ is CM with $\dim(R_\Delta)_\mathfrak{p} + \dim(R_\Delta/\mathfrak{p}) = \dim R_\Delta$ for any $\mathfrak{p} \in \text{Proj}\ (R_\Delta)$ by [HIO, Lemma 43.3], and so $X \cong \text{Proj}\ (R_\Delta)$ is CM and equidimensional. By using Lemma 3.2.2 one obtains (vi).

$(vi) \Rightarrow (i)$ Obvious.

The implications $(i) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (vi)$ may be proved similarly. $\square$

3.3 Applications

In this section we show several situations in which we can ensure the existence of Cohen-Macaulay coordinate rings for the blow-up scheme by using Theorem 3.2.3. First lemma provides sufficient conditions to have $\Gamma(\mathrm{X}, \mathcal{O}_\mathrm{X}) = k$ and $H^i(\mathrm{X}, \mathcal{O}_\mathrm{X}) = 0$ for all $0 < i < \overline{n} - 1$.

Lemma 3.3.1 *Assume $a^2_*(R) < 0$, $a_*(A) < 0$. Then $\Gamma(\mathrm{X}, \mathcal{O}_\mathrm{X}) = k$ and $H^i(\mathrm{X}, \mathcal{O}_\mathrm{X}) = 0$ for $0 < i < \overline{n} - 1$.*

Proof. Note that for any i, we have $H^i_{\mathcal{M}}(R)_{(0,0)} = H^i_{\mathcal{M}_2}(R)_{(0,0)} = 0$ and $H^i_{\mathcal{M}_1}(R)_{(0,0)} = H^i_{\mathfrak{m}}(A)_0 = 0$ by Proposition 2.1.18. Then, from the Mayer-Vietoris exact sequence associated to $\mathcal{M}_1$ and $\mathcal{M}_2$, we get $H^i_{R_+}(R)_{(0,0)} = 0$ for any i, so we are done. $\square$

As an immediate consequence we get:

Corollary 3.3.2 *Suppose that $a^2_*(R) < 0$, $a_*(A) < 0$. If X is Cohen-Macaulay equidimensional, then $k[(I^e)_c]$ is Cohen-Macaulay for $c \gg e \gg 0$.*

It is known that there are smooth projective varieties with no arithmetically Cohen-Macaulay embeddings (see for instance [Mat, Theorem 3.4]). Next we exhibit a situation where this implication is true.

Proposition 3.3.3 *Let X be the blow-up of $\mathbb{P}^{n-1}_k$ along a closed subscheme, where k has $\operatorname{char} k = 0$. Assume that X is smooth or with rational singularities. Then X is arithmetically Cohen-Macaulay.*

Proof. Let $\pi : X \to \mathbb{P}_k^{n-1}$ be the blow-up morphism. From [KKMS], we have that $\pi_* \mathcal{O}_{\mathrm{X}} = \mathcal{O}_{\mathbb{P}_k^{n-1}}$ and $R^j \pi_* \mathcal{O}_{\mathrm{X}} = 0$ for all $j > 0$. This implies that the Leray spectral sequence

$$E_2^{i,j} = H^i(\mathbb{P}_k^{n-1}, R^j \pi_* \mathcal{O}_{\mathrm{X}}) \Longrightarrow H^{i+j}(\mathrm{X}, \mathcal{O}_{\mathrm{X}})$$

degenerates. Therefore we have $\Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) = \Gamma(\mathbb{P}_k^{n-1}, \mathcal{O}_{\mathbb{P}_k^{n-1}}) = k$ and $H^i(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) = H^i(\mathbb{P}_k^{n-1}, \mathcal{O}_{\mathbb{P}_k^{n-1}}) = 0$ for all $i > 0$. Then the result follows from Corollary 3.2.5. $\square$

Assume that A is Cohen-Macaulay. S.D. Cutkosky and J. Herzog proved in [CH] that the Rees algebra has Cohen-Macaulay diagonals for locally complete intersection ideals and for ideals whose homogeneous localizations are strongly Cohen-Macaulay satisfying condition $(\mathcal{F}_1)$. In the first case, observe that $R_{A_\mathfrak{p}}(I_\mathfrak{p})$ is Cohen-Macaulay for any $\mathfrak{p} \in \mathrm{Proj}\ (A)$, while in the second one $R_{A_{(\mathfrak{p})}}(I_{(\mathfrak{p})})$ is Cohen-Macaulay for any $\mathfrak{p} \in \mathrm{Proj}\ (A)$. Next we want to study those examples.

Proposition 3.3.4 *The following are equivalent:*

(i) $R_{A_{x_i}}(I_{x_i})$ *is Cohen-Macaulay for all* $1 \le i \le n$.

(ii) $R_{A_{(x_i)}}(I_{(x_i)})$ *is Cohen-Macaulay for all* $1 \le i \le n$.

(iii) $R_{A_\mathfrak{p}}(I_\mathfrak{p})$ *is Cohen-Macaulay for all* $\mathfrak{p} \in \mathrm{Proj}\ (A)$.

(iv) $R_{A_{(\mathfrak{p})}}(I_{(\mathfrak{p})})$ *is Cohen-Macaulay for all* $\mathfrak{p} \in \mathrm{Proj}\ (A)$.

Proof. Set $\overline{R}_i = R_{A_{x_i}}(I_{x_i})$, $R_i = R_{A_{(x_i)}}(I_{(x_i)})$. We have already shown in the proof of Theorem 3.1.3 that there exists an isomorphism $\overline{R}_i \cong R_i[T, T^{-1}]$. Therefore, R_i is CM if and only if $\overline{R}_i$ is CM, and so the two first conditions are equivalent.

Now let us prove $(i) \Longleftrightarrow (iii)$. First assume (i), and for any prime ideal $\mathfrak{p} \in \mathrm{Proj}\ (A)$ let us take $x_i \notin \mathfrak{p}$. Note that $R_{A_\mathfrak{p}}(I_\mathfrak{p}) = \overline{R}_i \otimes_{A_{x_i}} (A_{x_i})_\mathfrak{p}$, and so $R_{A_\mathfrak{p}}(I_\mathfrak{p})$ is Cohen-Macaulay. Now assume (iii), and let us think $\overline{R}_i$ as a bigraded ring. Then, to prove (i), it is enough to show that for any homogeneous prime ideal $Q \in \mathrm{Spec}(\overline{R}_i)$, we have that $(\overline{R}_i)_Q$ is CM. Given such a Q, denote by $\mathfrak{q}A_{x_i} = Q \cap A_{x_i}$, where $\mathfrak{q} \in \mathrm{Spec}(A)$ is a homogeneous prime which does not contain x_i, that is, $\mathfrak{q} \in \mathrm{Spec}(A_{(x_i)}) \subset \mathrm{Proj}\ (A)$. Then we have $(\overline{R}_i)_Q = (R_{A_\mathfrak{q}}(I_\mathfrak{q}))_Q$, and so $(\overline{R}_i)_Q$ is CM.

Finally, let us prove $(ii) \iff (iv)$. Given $\mathfrak{p} \in \text{Proj }(A)$ and $x_i \notin \mathfrak{p}$, let $\mathfrak{q} = \mathfrak{p}A_{x_i} \cap A_{(x_i)} \in \text{Spec}(A_{(x_i)})$. Then, $(A_{(x_i)})_{\mathfrak{q}} = A_{(\mathfrak{p})}$ and so $R_{A_{(\mathfrak{p})}}(I_{(\mathfrak{p})}) = R_i \otimes_{A_{(x_i)}} (A_{(x_i)})_{\mathfrak{q}}$. Therefore, (ii) implies (iv). Now let us assume (iv). Given any homogeneous prime ideal $Q \in \text{Spec}(R_i)$, let $\mathfrak{q} = Q \cap A_{(x_i)} \in \text{Spec}(A_{(x_i)}) \subset \text{Proj }(A)$, and let $\mathfrak{p} \in \text{Proj }(A)$ such that $\mathfrak{p}A_{x_i} = \mathfrak{q}[x_i, x_i^{-1}]$. Since $(R_i)_Q = (R_{A_{(\mathfrak{p})}}(I_{(\mathfrak{p})}))_Q$, we have that $(R_i)_Q$ is Cohen-Macaulay, and so R_i is CM. □

Now we can prove that the Rees algebra of a homogeneous ideal I in a Cohen-Macaulay ring A satisfying any of the equivalent conditions above has Cohen-Macaulay diagonals. More generally, we have:

Theorem 3.3.5 *Assume that A is equidimensional and $R_{A_\mathfrak{p}}(I_\mathfrak{p})$ is Cohen-Macaulay for all $\mathfrak{p} \in \text{Proj }(A)$. Then $k[(I^e)_c]$ is Cohen-Macaulay for $c \gg 0$ relatively to $e \gg 0$ if and only if $H^i_\mathfrak{m}(A)_0 = 0$ for all $i < \overline{n}$.*

Proof. Given $P \in X$, let us denote by $\mathfrak{p} = P \cap A \in \text{Proj }(A)$. Then $R_A(I)_P = (R_{A_\mathfrak{p}}(I_\mathfrak{p}))_P$ is CM and so X is CM. Then, by Corollary 3.1.4 and the Leray spectral sequence

$$E_2^{i,j} = H^i(Y, R^j\pi_*\mathcal{O}_{\mathrm{X}}) \Longrightarrow H^{i+j}(\mathrm{X}, \mathcal{O}_{\mathrm{X}}),$$

we get $H^j(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) = H^j(\mathrm{Y}, \mathcal{O}_{\mathrm{Y}}) = H^{j+1}_\mathfrak{m}(A)_0$ for $0 < j < \overline{n} - 1$, and the exact sequence $0 \to H^0_\mathfrak{m}(A)_0 \to k \to \Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) = \Gamma(\mathrm{Y}, \mathcal{O}_{\mathrm{Y}}) \to H^1_\mathfrak{m}(A)_0 \to 0$, so we get the statement. □

Denote by E the exceptional divisor of the blow-up and by w_E its dualizing sheaf. The last result of the section shows that weaker assumptions on [CH, Lemma 2.1] are enough to ensure that the rings $k[(I^e)_c]$ are Cohen-Macaulay for $c \gg e \gg 0$.

Proposition 3.3.6 *Suppose that A is Cohen-Macaulay, X is a Cohen-Macaulay scheme, $\pi_*\mathcal{O}_E(m) = \widetilde{I}^m/\widetilde{I}^{m+1}$ for $m \geq 0$ and $R^i\pi_*\mathcal{O}_E(m)$=0 for $i > 0$ and $m \geq 0$. Then $k[(I^e)_c]$ is Cohen-Macaulay for $c >> 0$ relatively to $e >> 0$.*

Proof. $R^i\pi_*\mathcal{O}_{\mathrm{X}} = 0$ for $i > 0$ and $\pi_*\mathcal{O}_{\mathrm{X}} = \mathcal{O}_{\mathrm{Y}}$ by [CH, Lemma 2.1]. Then, from the Leray spectral sequence, we obtain $H^i(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) = H^i(\mathrm{Y}, \mathcal{O}_{\mathrm{Y}}) = H^{i+1}_\mathfrak{m}(A)_0 = 0$ for $0 < i < \overline{n} - 1$ and $\Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) = \Gamma(\mathrm{Y}, \mathcal{O}_{\mathrm{Y}}) = k$. Now, the proposition follows from Corollary 3.2.5. □

3.4 Cohen-Macaulay diagonals

Once we have studied the problem of the existence of Cohen-Macaulay diagonals of a Rees algebra, now we would like to study in more detail which diagonals are Cohen-Macaulay. This question has been totally answered only for complete intersection ideals in the polynomial ring [CHTV, Theorem 4.6]. Our approach to this problem will give us criteria to decide if a diagonal is Cohen-Macaulay, which will allow us to recover and extend the result in [CHTV] to any Cohen-Macaulay ring as well as to precise the Cohen-Macaulay diagonals for new families of ideals.

The first criterion gives necessary and sufficient conditions for a diagonal of a Cohen-Macaulay Rees algebra to have this property in the case where I is equigenerated. Namely,

Proposition 3.4.1 *Let $I \subset A$ be a homogeneous ideal generated by forms of degree d whose Rees algebra is Cohen-Macaulay. For any $c \geq de+1$, $k[(I^e)_c]$ is Cohen-Macaulay if and only if*

(i) $H^i_{\mathfrak{m}}(A)_0 = 0$, for $i < \overline{n}$.

(ii) $H^i_{\mathfrak{m}}(I^{es})_{cs} = 0$, for $i < \overline{n}$, $s > 0$.

Proof. First, recall that the assumptions on the local cohomology of A are necessary and sufficient conditions for the existence of Cohen-Macaulay diagonals (Theorem 2.3.13). Then, for any $c \geq de+1$ and $i < \overline{n}$, we have $H^i_m(k[(I^e)_c])_0 = 0$ by Theorem 3.2.3 and Remark 3.1.1.

On the other hand, by applying Proposition 2.1.18 and Proposition 2.1.19, for any $s < 0$ we have:

$$H^q_{\mathcal{M}_1}(R)_{(cs,es)} = H^q_{\mathfrak{m}_1}(R^{es})_{cs} = 0$$

$$H^q_{\mathcal{M}_2}(R)_{(cs,es)} = H^q_{\mathfrak{m}_2}(R_{cs-des})_{es} = 0$$

because $R^{es} = 0$ and $R_{cs-des} = 0$. Therefore, for any diagonal and any $i < \overline{n}$, $s < 0$, we get $H^i_m(k[(I^e)_c])_s = 0$ according to Proposition 2.1.3. The statement, then, follows from Corollary 2.3.5. $\square$

We may apply Proposition 3.4.1 to study in detail the following example considered by L. Robbiano and G. Valla in [RV].

Corollary 3.4.2 *Let $\{L_{ij}\}$ be a set of $d \times (d+1)$ homogeneous linear forms of a polynomial ring $A = k[X_1, \ldots, X_n]$, $i = 1, \ldots, d$; $j = 1, \ldots, d+1$, and let M be the matrix (L_{ij}). Let $I_t(M)$ be the ideal generated by the $t \times t$ minors of M and assume that $\mathrm{ht}\,(I_t(M)) \geq d - t + 2$ for $1 \leq t \leq d$. Denoting by $I = I_d(M)$, then $k[(I^e)_c]$ is Cohen-Macaulay for any $c \geq de + 1$.*

Proof. The ideal I is generated by $d+1$ forms of degree d, and the Rees algebra has a presentation of the form

$$R_A(I) = k[X_1, \ldots, X_n, Y_1, \ldots, Y_{d+1}]/(\phi_1, \ldots, \phi_d),$$

where the polynomial ring $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_{d+1}]$ is bigraded by $\deg(X_i) = (1, 0)$, $\deg(Y_j) = (d, 1)$, and $\phi_1, \ldots, \phi_d$ is a regular sequence in S with $\deg(\phi_l) = (d+1, 1)$ (see the proof of [RV, Theorem 5.11]). Then we have a bigraded minimal free resolution of the Rees algebra $R_A(I)$ as S-module given by the Koszul complex associated to $\phi_1, \ldots, \phi_d$:

$$0 \to F_d \to \cdots \to F_1 \to F_0 = S \to R_A(I) \to 0,$$

with $F_p = S(-(d+1)p, -p)^{\binom{d}{p}}$. By applying the functor $(\ )^e$ to this resolution, we have a graded free resolution of I^e over A:

$$0 \to F_p^e \to \cdots \to F_1^e \to F_0^e = S^e \to I^e \to 0,$$

with $p = \min\{e, d\}$, $F_p^e = A(-p - de)^{\rho_p^e}$ for certain $\rho_p^e \in \mathbb{Z}$, $\rho_p^e \neq 0$. The minimal graded free resolution of I^e is then obtained by picking out some terms, but in any case it must have length p because the Hilbert series of A/I^e is given by ([RV, Example 6.1])

$$H_{A/I^e}(z) = \frac{1 - \sum_{j=0}^{d} (-1)^j \binom{d}{j} \binom{d+e-j}{e-j} z^{de+j}}{(1-z)^n}$$

(note that z^{p+de} appears in the numerator). So by Theorem 1.3.4 we can compute the a_*-invariant of I^e and we get

$$a_*(I^e) = \begin{cases} de + e - n & \text{if } e < d \\ de + d - n & \text{if } e \geq d. \end{cases}$$

On the other hand, since $n \geq \mathrm{ht}\,(I_1(M)) \geq d + 1$, we have that $d \leq n - 1$, and so $a_*(I^e) < de$. Therefore, for any $c \geq de + 1$, $s \geq 1$, we have that $H_{\mathfrak{m}}^i(I^{es})_{cs} = 0$ for all i. So $k[(I^e)_c]$ is Cohen-Macaulay by Proposition 3.4.1. Furthermore, note that $a(k[(I^e)_c]) < 0$. $\square$

For arbitrary homogeneous ideals, we can also get a criterion for the Cohen-Macaulayness of the diagonals by means of the local cohomology of the powers of the ideal and the local cohomology of the graded pieces of the canonical module of the Rees algebra. More explicitly,

Theorem 3.4.3 *Let I be a homogeneous ideal in A generated by forms of degree $\leq d$ whose Rees algebra is Cohen-Macaulay. For any $c \geq de+1$, $k[(I^e)_c]$ is Cohen-Macaulay if and only if*

(i) $H^i_{\mathfrak{m}}(A)_0 = 0$ *for* $i < \overline{n}$.

(ii) $H^i_{\mathfrak{m}}(I^{es})_{cs} = 0$ *for* $i < \overline{n}$, $s > 0$.

(iii) $H^{\overline{n}-i+1}_{\mathfrak{m}}(K^{es})_{cs} = 0$ *for* $1 \leq i < \overline{n}$, $s > 0$.

Proof. As in the proof of Proposition 3.4.1, the assumptions on the local cohomology of A are necessary and sufficient conditions for the existence of Cohen-Macaulay diagonals. Then we have that $H^i_m(k[(I^e)_c])_0 = 0$ for $i < \overline{n}$.

Since R is Cohen-Macaulay, we have that K_R is Cohen-Macaulay with $a^2_*(K_R) = 0$. Therefore, for any $s > 0$, $1 \leq i < \overline{n}$, $H^i_m(k[(I^e)_c])_{-s} = H^{\overline{n}-i+1}_{\mathfrak{m}}(K^{es})_{cs}$ by Proposition 3.1.6. Moreover, note that $H^0_m(k[(I^e)_c])_{-s} = 0$ for any $s > 0$. Then the statement follows from Corollary 2.3.5. □

Let us denote by $G = G_A(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ the form ring of I with the natural bigrading as a quotient of the Rees algebra. For ideals whose form ring is quasi-Gorenstein, we may get necessary and sufficient conditions for a diagonal to be Cohen-Macaulay only in terms of the powers of the ideal.

Corollary 3.4.4 *Let I be a homogeneous ideal in A generated by forms of degree $\leq d$. Assume that the Rees algebra is Cohen-Macaulay and the form ring is quasi-Gorenstein. Let $a = -a^2(G_A(I))$, $b = -a(A)$. For any $c \geq de+1$, $k[(I^e)_c]$ is Cohen-Macaulay if and only if*

(i) $H^i_{\mathfrak{m}}(A)_0 = 0$, *for* $i < \overline{n}$.

(ii) $H^i_{\mathfrak{m}}(I^{es})_{cs} = 0$, *for* $i < \overline{n}$, $s > 0$.

(iii) $H^i_{\mathfrak{m}}(I^{es-a+1})_{cs-b} = 0$, *for* $1 < i \leq \overline{n}$, $s > 0$.

Proof. Under these assumptions K_R has the expected form, that is, there is a bigraded isomorphism

$$K_{R_A(I)} \cong \bigoplus_{(l,m),\, m \geq 1} [I^{m-a+1}]_{l-b}$$

(see Corollary 4.1.7 for more details about the isomorphism). Then, for any $s > 0$ we have $K^{es} \cong I^{es-a+1}(-b)$, and now the result follows from Theorem 3.4.3. □

We can use Corollary 3.4.4 to precise the Cohen-Macaulay diagonals for a complete intersection ideal of a Cohen-Macaulay ring. In particular, this gives a new proof of [CHTV, Theorem 4.6] where the case $A = k[X_1, \ldots, X_n]$ was studied.

Proposition 3.4.5 *Let I be a complete intersection ideal of a Cohen-Macaulay ring A minimally generated by r forms of degrees $d_1, \ldots, d_r$. Set $u = \sum_{i=1}^{r} d_i$. For any $c \geq de+1$, $k[(I^e)_c]$ is Cohen-Macaulay if and only if $c > (e-1)d + u + a(A)$.*

Proof. From the bigraded isomorphism $G_A(I) \cong A/I[Y_1, \ldots, Y_r]$, with $\deg(Y_j) = (d_j, 1)$, it is easy to prove by induction on e that the a_*-invariant of A/I^e is:

$$a_*(A/I^e) = a(A/I^e) = (e-1)d + u + a(A).$$

On the other hand, we also have that $H_{\mathfrak{m}}^{\overline{n}-r}(A/I^e)_s \neq 0$, for all $s \leq a(A/I^e) = (e-1)d + u + a(A)$ (see Lemma 5.2.19).

Let $\Delta = (c, e)$ be a diagonal with $c \geq de+1$. Since $a^2(G) = -\mathrm{ht}\,(I) = -r$, by Corollary 3.4.4 we have that $k[(I^e)_c]$ is Cohen-Macaulay if and only if $cs > a(A/I^{es})$ and $cs + a(A) > a(A/I^{es-r+1})$ for all $s > 0$. The first condition is equivalent to $(c-de)s > u-d+a(A)$ for all $s > 0$, that is, $c-de > u-d+a(A)$. The other one is equivalent to $(c-de)s > u-dr$ for all $s > 0$, and this always holds because $u - dr \leq 0$. □

Until now we have given criteria to decide if a diagonal $k[(I^e)_c]$ is Cohen-Macaulay once we know the local cohomology of the powers of I, and the local cohomology of the graded pieces of the canonical module of the Rees algebra. We will apply them in Chapter 5, Section 2, after computing the local cohomology of the powers of certain families of ideals.

The following result shows the behaviour of the a_*-invariant for the graded pieces of any finitely generated bigraded S-module, so in particular for the powers of an ideal and the pieces of the canonical module by applying it to the Rees algebra and its canonical module respectively. This fact has been also obtained independently by S.D. Cutkosky, J. Herzog and N. V. Trung [CHT] and V. Kodiyalam [Ko2] by different methods (see Chapter 5 for more details).

Theorem 3.4.6 *Let L be a finitely generated bigraded S-module. Then there exists α such that for any e*

$$a_*(L^e) \leq de + \alpha.$$

Proof. Let $e_0 = a_*^2(L)$. By Proposition 2.1.18, $H^i_{\mathcal{M}_2}(L)_{(c,e)} = 0$ for $i \geq 0$, $e > e_0$. Then, by Proposition 2.1.3 and Proposition 2.1.18, we have that for any $c \geq de+1$, $e > e_0$, $i \geq 0$, there are isomorphisms

$$H^i_m(L_\Delta)_1 \cong H^i_{\mathcal{M}_1}(L)_{(c,e)} \cong H^i_{\mathfrak{m}}(L^e)_c.$$

On the other hand, from Corollary 2.1.12 there exist positive integers e_1, α_1 such that $H^i_m(L_\Delta)_s \cong H^{i+1}_{\mathcal{M}}(L)_{(cs,es)}$ for $s \neq 0$, $e > e_1$, $c > de + \alpha_1$. Therefore, we have $H^i_{\mathfrak{m}}(L^e)_c = 0$ for $e > \max\{e_0, e_1\}$, $c > de + \alpha_1$, $i \geq 0$. This proves the statement. $\square$

Next we will show how to obtain a family of Cohen-Macaulay diagonals from the bound on the shifts in the bigraded minimal free resolution of the Rees algebra given by Theorem 1.3.4. To begin with, let us study the bigraded a-invariant of the Rees algebra.

Lemma 3.4.7 *(i) $a^1(R) \leq a(A)$.*

(ii) If R is Cohen-Macaulay and $a^2(G) < -1$, then $a^1(R) = a(A)$.

Proof. By setting $R_{++} = \bigoplus_{j>0} R_{(i,j)}$, we have the following bigraded exact sequences:

$$0 \to R_{++} \to R \to A \to 0$$

$$0 \to R_{++}(0,1) \to R \to G \to 0.$$

For each (i,j), we get exact sequences:

$$\ldots \to H^{\overline{n}}_{\mathcal{M}}(A)_{(i,j)} \to H^{\overline{n}+1}_{\mathcal{M}}(R_{++})_{(i,j)} \to H^{\overline{n}+1}_{\mathcal{M}}(R)_{(i,j)} \to 0 \qquad (1)$$

$$\ldots \to H^{\overline{n}}_{\mathcal{M}}(G)_{(i,j)} \to H^{\overline{n}+1}_{\mathcal{M}}(R_{++})_{(i,j+1)} \to H^{\overline{n}+1}_{\mathcal{M}}(R)_{(i,j)} \to 0 \qquad (2)$$

Note that $A_{(i,j)} = 0$ if $j \neq 0$ and so $H^{\overline{n}}_{\mathcal{M}}(A)_{(i,j)} = 0$ if $j \neq 0$.

We want to determine $a^1(R) = \max\{i \mid \exists j : H^{\overline{n}+1}_{\mathcal{M}}(R)_{(i,j)} \neq 0\}$. Suppose $H^{\overline{n}+1}_{\mathcal{M}}(R)_{(i,j)} \neq 0$. Since $a^2(R) = -1$, we have $j \leq -1$. Then, from (2), we get $H^{\overline{n}+1}_{\mathcal{M}}(R_{++})_{(i,j+1)} \neq 0$. If $j+1 < 0$, from (1) we obtain $H^{\overline{n}+1}_{\mathcal{M}}(R)_{(i,j+1)} \cong H^{\overline{n}+1}_{\mathcal{M}}(R_{++})_{(i,j+1)} \neq 0$. By repeating this argument,

we obtain $H_{\mathcal{M}}^{\overline{n}+1}(R_{++})_{(i,0)} \neq 0$ and, since $H_{\mathcal{M}}^{\overline{n}+1}(R)_{(i,0)} = 0$, from (1) we get $H_{\mathfrak{m}}^{\overline{n}}(A)_i = H_{\mathcal{M}}^{\overline{n}}(A)_{(i,0)} \neq 0$. Thus $i \leq a(A)$, and then it follows that $a^1(R) \leq a(A)$.

Assume now that R is Cohen-Macaulay and $a^2(G) < -1$. From (2), we have that if $H_{\mathcal{M}}^{\overline{n}+1}(R)_{(a(A),-1)} = 0$ then $H_{\mathcal{M}}^{\overline{n}+1}(R_{++})_{(a(A),0)} = 0$. Since R is Cohen-Macaulay, from (1) we get $H_{\mathfrak{m}}^{\overline{n}}(A)_{a(A)} = H_{\mathcal{M}}^{\overline{n}}(A)_{(a(A),0)} = 0$, which is a contradiction. □

Remark 3.4.8 Note that in the proof of the Lemma 3.4.7 (ii) it is enough to assume $H_{\mathcal{M}}^{\overline{n}}(G)_{(a(A),-1)} = 0$ and $H_{\mathcal{M}}^{\overline{n}}(R)_{(a(A),0)} = 0$.

Remark 3.4.9 Let us consider the group morphism $\psi : \mathbb{Z}^2 \to \mathbb{Z}$ defined by $\psi(i,j) = i+j$. By Lemma 1.2.3, $H_{\mathcal{M}}^{\overline{n}+1}(R^{\psi})_l = \bigoplus_{i+j=l} H_{\mathcal{M}}^{\overline{n}+1}(R)_{(i,j)}$. Then, by applying Lemma 3.4.7 we get $a(R^{\psi}) \leq a(A) - 1$. If R is Cohen-Macaulay and $a^2(G) < -1$, we have proved $H_{\mathcal{M}}^{\overline{n}+1}(R)_{(a(A),-1)} \neq 0$ and so $a(R^{\psi}) = a(A) - 1$.

We can use the upper bound for the bigraded a-invariant of the Rees algebra found in Lemma 3.4.7 to get bounds for the shifts (a,b) in its resolution. Namely,

Lemma 3.4.10 *Let I be an ideal of A generated by r forms in degrees $d_1 \leq \ldots \leq d_r$ whose Rees algebra is Cohen-Macaulay. Set $u = \sum_{j=1}^{r} d_j$. Let*

$$0 \to D_m \to \ldots \to D_1 \to D_0 = S \to R_A(I) \to 0$$

be the minimal bigraded free resolution of $R_A(I)$ over S. Given $p \geq 1$ and $(a,b) \in \Omega_p$, we have

(i) $a \leq 0$, $b \leq 0$, $a \leq d_1 b$.

(ii) $-a - b \leq u + \overline{n} + a(A) + p$.

(iii) $-a \leq u + \overline{n} + a(A) + p - (r-1)$. *In particular,* $-a \leq u + n + a(A)$.

(iv) $-b < r$.

Proof. It is clear that $a \leq 0$, $b \leq 0$, $a \leq d_1 b$. Also note that $m = \text{proj.dim}_S R = n + r - \overline{n} - 1$. To prove (ii), let us consider the morphism $\psi : \mathbb{Z}^2 \to \mathbb{Z}$ defined by $\psi(i,j) = i + j$, and note that $S(a,b)^{\psi} = S^{\psi}(a+b)$. Applying the functor $(\ \ )^{\psi}$ to the resolution, we get a $\mathbb{Z}$-graded minimal free

resolution of R^ψ over S^ψ. Moreover $a(S^\psi) = -n-u-r$ and $a(R^\psi) \leq a(A)-1$ (see Remark 3.4.9). Given $(a,b) \in \Omega_p$, from Theorem 1.3.4 we get:

$$\begin{aligned} -a-b &\leq \max\{\, -\alpha-\beta \mid (\alpha,\beta) \in \Omega_p\} \leq \\ &\leq \max\{\, -\alpha-\beta \mid (\alpha,\beta) \in \Omega_m\} + p - m = \\ &= a(R^\psi) - a(S^\psi) + p - m \leq u + a(A) + \overline{n} + p. \end{aligned}$$

To prove (iii), observe that by Theorem 1.3.4 we have

$$\begin{aligned} -a &\leq \max\{-\alpha \mid (\alpha,\beta) \in \Omega_p\} \leq \\ &\leq \max\{\, -\alpha \mid (\alpha,\beta) \in \Omega_m\} + p - m = \\ &= a^1(R) - a^1(S) + p - m \leq u + a(A) + \overline{n} - r + 1 + p. \end{aligned}$$

Finally, by using Theorem 1.3.4 we also obtain:

$$\begin{aligned} -b &\leq \max\{\, -\beta \mid (\alpha,\beta) \in \Omega_p\} \leq \\ &\leq \max\{\, -\beta \mid (\alpha,\beta) \in \Omega_m\} = \\ &= a^2(R) - a^2(S) = -1 + r\ , \end{aligned}$$

so (iv) is proved. □

Remark 3.4.11 When I is a complete intersection ideal of the polynomial ring $A = k[X_1, \ldots, X_n]$, all the shifts in the resolution may be explicitly computed. In fact, by the Eagon-Northcott complex the shifts $(a,b) \in \Omega_p$ are of the type:

$$a = -d_{j_1} - \ldots - d_{j_{p+1}}, \ \ b = -m$$

where $1 \leq j_1 \leq \ldots \leq j_{p+1} \leq r$, $1 \leq m \leq p$ (see [CHTV, Lemma 4.1]). Note that b takes all the values between $-r$ and 0 and the bounds of Lemma 3.4.10 (ii), (iii) are sharp for $p = r-1$.

Now we are ready to determine a family of diagonals of the Rees algebra with the Cohen-Macaulay property when the Rees algebra is Cohen-Macaulay. Namely,

Theorem 3.4.12 *Let $I \subset A$ be a homogeneous ideal generated by r forms of degrees $d_1 \leq \ldots \leq d_r = d$. Assume that $H^i_{\mathfrak{m}}(A)_0 = 0$ for all $i < \overline{n}$. Set $u = \sum_{j=1}^r d_j$. If the Rees algebra is Cohen-Macaulay, then*

(i) $k[(I^e)_c]$ *is Cohen-Macaulay for* $c > \max\{d(e-1)+u+a(A), d(e-1)+u-d_1(r-1)\}$.

(ii) If I *is equigenerated by forms of degree* d, $k[(I^e)_c]$ *is Cohen-Macaulay for* $c > d(e-1+l)+a(A)$.

Proof. We have already shown that the assumptions on the local cohomology of A imply that $H^i_m(k[(I^e)_c])_0 = 0$ for $i < \overline{n}$. Now, let us consider the bigraded minimal free resolution of R over S:

$$0 \to D_m \to \ldots \to D_0 = S \to R \to 0\ ,$$

where $D_p = \bigoplus_{(a,b)\in\Omega_p} S(a,b)$. From Remark 2.1.11, recall that if we define

$$X^\Delta = \bigcup_{(a,b)\in\Omega_R}\{\ s\in\mathbb{Z} \mid \tfrac{-b}{e} \le s \le \tfrac{bd-a-n}{c-ed}\ \},$$

$$Y^\Delta = \bigcup_{(a,b)\in\Omega_R}\{\ s\in\mathbb{Z} \mid \tfrac{(b+r)d-u-a}{c-ed} \le s \le \tfrac{-b-r}{e}\ \}\,,$$

then we have $H^i_m(k[(I^e)_c])_s = H^{i+1}_{\mathcal{M}}(R)_{(cs,es)} = 0$ for $i < \overline{n}$, $s \notin X^\Delta \cup Y^\Delta$. Therefore, $k[(I^e)_c]$ is Cohen-Macaulay for any diagonal $\Delta = (c,e)$ such that $X^\Delta \cup Y^\Delta \subset \{0\}$. Since $b \le 0$, any $s \in X^\Delta$ satisfies $s \ge 0$. If $b \le -1$, then $bd-a-n \le -d+u+a(A)$ by Lemma 3.4.10. If $b=0$, then note that $[\mathbf{D}.]_0$ is a graded minimal free resolution of A over S_1 with $[D_p]_0 = \bigoplus_{(a,0)\in\Omega_p} S_1(a)$, so $bd-a-n = -a-n \le a(A)$ by Theorem 1.3.4. Therefore, by taking $c > (e-1)d+u+a(A)$, we have $\frac{bd-a-n}{c-ed} < 1$ and so $X^\Delta \subset \{0\}$. On the other hand, any shift $(a,b) \in \Omega_R$ satisfies $b > -r$ by Lemma 3.4.10, so if $s \in Y^\Delta$ then $s \le -1$. By taking $c > d(e-1)+u-d_1(r-1)$, one can check $\frac{(b+r)d-u-a}{c-ed} > -1$, so $Y^\Delta = \emptyset$. This proves (i).

Now, let us assume that I is generated in degree d. From the proof of Proposition 3.4.1, we have that $H^i_m(k[(I^e)_c])_s = 0$ for $i < \overline{n}$, $s < 0$. So it is just enough to study the positive components of these local cohomology modules. Tensoriazing by $k(T)$ we may assume that the field k is infinite. Then, since the fiber cone $F_{\mathfrak{m}}(I)$ of I is a k-algebra generated by homogeneous elements in degree $(d,1)$, there exists a minimal reduction J of I generated by l forms of degree d. Now, by considering the polynomial ring $S = k[X_1,\ldots,X_n,Y_1,\ldots,Y_l]$, we have a natural epimorhism $S \to R_A(J)$. Then $R_A(J)$ is a finitely generated bigraded S-module, and so $R_A(I)$ because it is a finitely generated $R_A(J)$-module. Then we may consider the bigraded minimal free resolution of $R_A(I)$ over S, and it suffices to check that the sets

X^Δ and Y^Δ associated to this resolution do not have positive integers for $c > d(e-1+l) + a(A)$. $\square$

In the case where $A = k[X_1, \ldots, X_n]$ we can improve the bounds slightly. More explicitly,

Theorem 3.4.13 *Let $I \subset A = k[X_1, \ldots, X_n]$ be a homogeneous ideal generated by r forms of degrees $d_1 \leq \ldots \leq d_r$. Assume that the Rees algebra $R_A(I)$ is Cohen-Macaulay. Then, by defining*

$$\alpha = \min\{d(e-1) + u - n, e(u-n)\}\ ,$$

$$\beta = \min\{d(e-1) + u - d_1(r-1), e(u-d_1)\}\ ,$$

we have that $k[(I^e)_c]$ is Cohen-Macaulay for all $c > \max\{de, \alpha, \beta\}$.

Proof. Note that the first homomorphism in the resolution of the Rees algebra is:

$$\begin{array}{ccc} D_0 = S & \longrightarrow & R_A(I) \\ X_i & \mapsto & X_i \\ Y_j & \mapsto & f_j t\ , \end{array}$$

so any shift $(a, b) \in \Omega_p$, with $p \geq 1$, satisfies $b < 0$. Note that if $\frac{bd-a-n}{c-ed} < \frac{-b}{e}$ for all $(a, b) \in \Omega_p$, $p \geq 1$, then X^Δ is empty. This condition is equivalent to $e(-a-n) < -bc$. Since $e(-a-n) \leq e(-n-u)$ by Lemma 3.4.10 and $-bc \geq c$, it suffices to take $c > e(u-n)$ to get this condition. Similarly, if $c > e(u-d_1)$ then $Y^\Delta = \emptyset$ and we are done. $\square$

Remark 3.4.14 With the notation above, note that $\alpha = e(u-n)$ if and only if $u-d-n < 0$, and $\beta = e(u-d_1)$ if and only if $u-d-d_1 < 0$ and $e > \frac{u-d-d_1(r-1)}{u-d_1-d}$. For instance, if $u-d < n$ then $k[(I^e)_c]$ is Cohen-Macaulay for all $c > \max\{de, \beta\}$.

We finish this section with an application of Corollary 3.1.4 to the study of the $(n-1)$-folds obtained from $\mathbb{P}_k^{n-1}$ by blowing-up a finite set of distinct points. Let $P_1, \ldots, P_s \in \mathbb{P}_k^{n-1}$ be distinct points, and for each $i = 1, \ldots, s$, denote by $\mathcal{P}_i \subset A = k[X_1, \ldots, X_n]$ the homogeneous prime ideal which corresponds to P_i. Let us consider the ideal of fat points $I = \mathcal{P}_1^{m_1} \cap \ldots \cap \mathcal{P}_s^{m_s}$, with $m_1, \ldots, m_s \in \mathbb{Z}_{\geq 1}$. Next we study the embeddings of the blow-up of $\mathbb{P}_k^{n-1}$ along $\mathcal{I}$ via the linear systems $(I^e)_c$, whenever these linear systems are very ample, slightly extending [GGP, Theorem 2.4] where only the divisors (I_c) were considered.

Theorem 3.4.15 *Let $I \subset A = k[X_1, \ldots, X_n]$ be an ideal of fat points, where k is a field with characteristic 0. Then:*

(i) $k[(I^e)_c]$ is Cohen-Macaulay if and only if $H^i_{\mathfrak{m}}(I^{es})_{cs} = 0$ for any $s > 0$, $i < n$.

(ii) For $c > \operatorname{reg}(I)\, e$, $k[(I^e)_c]$ is Cohen-Macaulay with $a(k[(I^e)_c]) < 0$. In particular, $\operatorname{reg}(k[(I^e)_c]) < n - 1$.

Proof. Let X be the blow-up of the projective space $\mathbb{P}^{n-1}_k$ along $\mathcal{I}$. Assume that I is generated by forms in degree $\leq d$. Then we have shown that $\mathcal{L}^e \otimes \mathcal{M}^c$ is very ample if $c > de$. Therefore, for any $s < 0$, $i < n-1$, $c > de$, we have that $H^i(X, \mathcal{L}^{es} \otimes \mathcal{M}^{cs}) = 0$ by the Kodaira vanishing theorem (see for instance [Har, Remark III.7.5]). Then, $H^i_m(k[(I^e)_c])_s = 0$ for $i < n$, $s < 0$ by Remark 3.1.1.

On the other hand, from Proposition 3.3.3 we get $\Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) = k$ and $H^i(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) = 0$ for all $i > 0$. Then, according to Remark 3.1.1, we have $H^i_m(k[(I^e)_c])_0 = 0$ for any i.

Finally, note that for a given $\mathfrak{p} \in \operatorname{Proj}(A)$ we have:

$$I_{\mathfrak{p}} = \begin{cases} A_{\mathfrak{p}} & \text{if } \mathfrak{p} \notin \{\mathcal{P}_1, \ldots, \mathcal{P}_s\} \\ \mathcal{P}_i^{m_i} A_{\mathcal{P}_i} & \text{if } \mathfrak{p} = \mathcal{P}_i \ . \end{cases}$$

In both cases, $R_{A_{\mathfrak{p}}}(I_{\mathfrak{p}})$ is Cohen-Macaulay. So, according to Corollary 3.1.5, we have that for any $s > 0$ and $c \geq de + 1$ there is an exact sequence

$$0 \to H^0_m(k[(I^e)_c])_s \to (I^{es})_{cs} \to (I^{es})^*_{cs} \to H^1_m(k[(I^e)_c])_s \to 0$$

and isomorphisms $H^i_m(k[(I^e)_c])_s \cong H^i_{\mathfrak{m}}(I^{es})_{cs}$ for $i > 1$. Therefore, we immediately get (i). From [GGP, Theorem 1.1] or [Cha, Theorem 6], we have $a_*(I^e) \leq \operatorname{reg}(I^e) \leq e \operatorname{reg}(I)$. Then, by taking $c > \operatorname{reg}(I)e$, we have $H^i_m(k[(I^e)_c])_s = 0$ for any $s > 0$. So $k[(I^e)_c]$ is Cohen-Macaulay with $a(k[(I^e)_c]) < 0$. $\square$

3.5 Linear bounds

S.D. Cutkosky and J. Herzog [CH] studied sufficient conditions for the existence of a constant f satisfying that the rings $k[(I^e)_c]$ are Cohen-Macaulay for all $c \geq ef$ and $e > 0$, that is, for the existence of a linear bound on c and

e ensuring that $k[(I^e)_c]$ is Cohen-Macaulay. Note that, according to Theorem 3.4.12, this holds for any homogeneous ideal in a Cohen-Macaulay ring whose Rees algebra is Cohen-Macaulay. Our first purpose is to show that this also holds under the weaker assumption that $R_{A_{\mathfrak{p}}}(I_{\mathfrak{p}})$ is Cohen-Macaulay for any $\mathfrak{p} \in \text{Proj }(A)$. This would recover for instance locally complete intersection ideals.

Let $K = K_R = \bigoplus_{(i,j)} K_{(i,j)}$ be the canonical module of $R = R_A(I)$, and let K^e be the graded A-module $K^e = \bigoplus_i K_{(i,e)}$. Then we have

Theorem 3.5.1 *Assume that $R_{A_{\mathfrak{p}}}(I_{\mathfrak{p}})$ is Cohen-Macaulay for all $\mathfrak{p} \in$ Proj (A). Then $\pi_*(w_X \otimes \mathcal{L}^e) = \widetilde{K^e}$ and $R^j\pi_*(w_X \otimes \mathcal{L}^e) = 0$ for $e > 0$, $j > 0$.*

Proof. Let $A_i = A_{(x_i)}$, $I_i = I_{(x_i)}$, $R_i = A_i[I_it]$ and $K_i = K_R \otimes R_i$. Let us consider the affine cover $\{Y_i : 1 \leq i \leq n\}$ of Y, where $Y_i = Y - V_+(x_i) \cong \text{Spec}(A_i)$. Denote by $X_i = \pi^{-1}Y_i = \text{Proj }(R_i)$. Then, for a given j and $e > 0$ we have that $R^j\pi_*(w_X \otimes \mathcal{L}^e) = 0$ if and only if $R^j\pi_*(w_X \otimes \mathcal{L}^e) \mid Y_i = 0$ for all $1 \leq i \leq n$. Furthermore, we have a diagram

$$\begin{array}{ccc} X_i = \text{Proj}(R_i) & \hookrightarrow & X = \text{Proj }^2(R) \\ \pi' \downarrow & & \pi \downarrow \\ Y_i = \text{Spec}(A_i) & \hookrightarrow & Y = \text{Proj}(A) \end{array}$$

Now, by Corollary III.8.2 and Proposition III.8.5 of [Har], for any $e > 0$ and $j > 0$ we have

$$R^j\pi_*(w_X \otimes \mathcal{L}^e) \mid Y_i = R^j\pi'_*((w_X \otimes \mathcal{L}^e) \mid X_i) = H^j(X_i, (w_X \otimes \mathcal{L}^e) \mid \mathrm{X}_i)^{\sim}.$$

Since $(w_X \otimes \mathcal{L}^e) \mid X_i = \widetilde{K_i(e)}$, we have reduced the problem to show that $H^{j+1}_{(R_i)_+}(K_i)_e = 0$. Similarly, $\pi_*(w_X \otimes \mathcal{L}^e) = \widetilde{K^e}$ if $H^0_{(R_i)_+}(K_i)_e = H^1_{(R_i)_+}(K_i)_e = 0$.

Denote by $\overline{R}_i = R_{A_{x_i}}(I_{x_i})$. Tensoriazing by $\overline{R}_i$, we have

$$H^j_{(\overline{R}_i)_+}(K \otimes \overline{R}_i)_e = H^j_{(R_i)_+}(K_i)_e[T, T^{-1}],$$

so it is enough to show that $H^j_{(\overline{R}_i)_+}(K \otimes \overline{R}_i)_e = 0$ for any i, j and $e > 0$. Let $\mathfrak{q} \in \text{Spec}(A_{x_i})$ be a homogeneous prime, and let $\mathfrak{p} \in \text{Proj }(A)$ be such that $\mathfrak{q} = \mathfrak{p}A_{x_i}$. Denote by $B = R_{A_{\mathfrak{p}}}(I_{\mathfrak{p}})$. Then

$$[H^j_{(\overline{R}_i)_+}(K \otimes \overline{R}_i)_e]_{\mathfrak{q}} = [H^j_{(\overline{R}_i)_+}(K \otimes \overline{R}_i)_{\mathfrak{q}}]_e = [H^j_{B_+}(K \otimes_A A_{\mathfrak{p}})]_e.$$

By taking into account that B is Cohen-Macaulay, standard arguments allow to check that $K \otimes_A A_{\mathfrak{p}} = K_B$ or $K \otimes_A A_{\mathfrak{p}} = 0$. In any case, we have that $[H^j_{B_+}(K \otimes_A A_{\mathfrak{p}})]_e = 0$ for any j and $e > 0$, so we are done. $\square$

From this result we can obtain a simple criterion for having a linear bound for the Cohen-Macaulay property. First, let us notice the following interesting fact.

Proposition 3.5.2 *Let A be an equidimensional ring. Assume $R_{A_{\mathfrak{p}}}(I_{\mathfrak{p}})$ is Cohen-Macaulay for all $\mathfrak{p} \in \mathrm{Proj}\ (A)$. Then, for any $c \geq de+1$ and $e > 0$:*

(i) For $s > 0$, there is an exact sequence

$$0 \to H^0_m(k[(I^e)_c])_s \to (I^{es})_{cs} \to (I^{es})^*_{cs} \to H^1_m(k[(I^e)_c])_s \to 0$$

and isomorphisms $H^i_m(k[(I^e)_c])_s \cong H^i_{\mathfrak{m}}(I^{es})_{cs}$ for $i > 1$.

(ii) For $s > 0$, $1 \leq i \leq \overline{n}-1$, $H^i_m(k[(I^e)_c])_{-s} \cong H^{\overline{n}-i+1}_{\mathfrak{m}}(K^{es})_{cs}$.

Proof. The first part of the statement follows directly from Corollary 3.1.5. To prove (ii), let $s > 0$, $i \geq 1$. Then

$$\begin{aligned} H^i_{\mathfrak{m}}(k[(I^e)_c])_{-s} &\cong H^{i-1}(X, \mathcal{L}^{-es} \otimes \mathcal{M}^{-cs}) \quad \text{by Remark 3.1.1} \\ &= H^{\overline{n}-i}(X, w_X \otimes \mathcal{L}^{es} \otimes \mathcal{M}^{cs}) \quad \text{by Serre's duality} \\ &= H^{\overline{n}-i}(Y, \pi_*(w_X \otimes \mathcal{L}^{es}) \otimes \mathcal{M}^{cs}) \quad \text{by Theorem 3.5.1} \\ &= H^{\overline{n}-i}(Y, \widetilde{K^{es}}(cs)) \quad \text{by Theorem 3.5.1} \\ &= H^{\overline{n}-i+1}_{\mathfrak{m}}(K^{es})_{cs}. \ \square \end{aligned}$$

Theorem 3.5.3 *Assume that A is an equidimensional ring with $H^i_{\mathfrak{m}}(A)_0 = 0$ for $i < \overline{n}$. If I is a homogeneous ideal of A such that $R_{A_{\mathfrak{p}}}(I_{\mathfrak{p}})$ is Cohen-Macaulay for all $\mathfrak{p} \in \mathrm{Proj}\ (A)$, then there exists α such that $k[(I^e)_c]$ is Cohen-Macaulay for $c \geq de + \alpha$, $e > 0$.*

Proof. From Proposition 3.3.5 we have that $k[(I^e)_c]$ is Cohen-Macaulay for $c \gg e \gg 0$. So, in particular, by Theorem 3.2.3 and Remark 3.1.1 we have that $H^i_m(k[(I^e)_c])_0 = 0$ for any $c \geq de+1$, $i < \overline{n}$. On the other hand, according to Theorem 3.4.6, there exists $\alpha > 0$ such that $a_*(I^e) < de+\alpha$, $a_*(K^e) < de+\alpha$, for all e. Then, $k[(I^e)_c]$ is Cohen-Macaulay for any $c \geq de+\alpha$ by Proposition 3.5.2. $\square$

In particular, we can recover Corollary 4.2 and Corollary 4.4 in [CH]. Furthermore, note that the bound has been improved slightly.

Corollary 3.5.4 *Let I be a locally complete intersection ideal in a Cohen-Macaulay ring A. Then there exists α such that such that $k[(I^e)_c]$ is Cohen-Macaulay for any $c \geq de + \alpha$ and $e > 0$.*

Corollary 3.5.5 *Let I be a homogeneous ideal such that $I_{(\mathfrak{p})}$ is a strongly Cohen-Macaulay ideal with $\mu(I_{(\mathfrak{p})}) \leq \operatorname{ht}(\mathfrak{p})$ for any prime ideal $\mathfrak{p} \supseteq I$. Then there exists α such that such that $k[(I^e)_c]$ is Cohen-Macaulay for any $c \geq de+\alpha$ and $e > 0$.*

We can also characterize the existence of linear bounds for the Cohen-Macaulay property of the rings $k[(I^e)_c]$ by means of the local cohomology modules of the Rees algebra and its canonical module. Namely,

Proposition 3.5.6 *Assume that there exist c, e such that $k[(I^e)_c]$ is Cohen-Macaulay. Then the following are equivalent*

(i) *There exists f such that $k[(I^e)_c]$ is Cohen-Macaulay for $c \geq ef$, $e > 0$.*

(ii) *There exists f such that $H^i_{R_+}(R)_{(c,e)} = 0$, $H^{\overline{n}-i+1}_{R_+}(K_R)_{(c,e)} = 0$, for $i < \overline{n}$, $c \geq ef$ and $e > 0$.*

(iii) *There exists f such that $H^i_{\mathcal{M}_2}(R)_{(c,e)} = 0$, $H^{\overline{n}-i+1}_{\mathcal{M}_2}(K_R)_{(c,e)} = 0$, for $i < \overline{n}$, $c \geq ef$ and $e > 0$.*

Proof. From Proposition 2.1.2, we have $H^i_m(k[(I^e)_c])_s = H^i_{R_+}(R)_{(cs,es)}$ for any $s > 0$. Moreover, since X is Cohen-Macaulay and equidimensional we also have that $H^i_m(k[(I^e)_c])_{-s} = H^{\overline{n}-i+1}_{R_+}(K_R)_{(cs,es)}$ for $1 \leq i < \overline{n}$ and $s > 0$ according to Proposition 3.1.6. Therefore two first conditions are equivalent.

To prove $(ii) \iff (iii)$, first we will show that there exists $\overline{f}$ such that for all i, $e > 0$, $c \geq e\overline{f}$ it holds

$$H^i_{\mathcal{M}}(R)_{(c,e)} = H^i_{\mathcal{M}_1}(R)_{(c,e)} = 0,$$

$$H^i_{\mathcal{M}}(K_R)_{(c,e)} = H^i_{\mathcal{M}_1}(K_R)_{(c,e)} = 0.$$

Then, from the Mayer-Vietoris exact sequence, we get that for any i, $e > 0$, $c \geq e\overline{f}$,

$$H^i_{R_+}(R)_{(c,e)} = H^i_{\mathcal{M}_2}(R)_{(c,e)},$$

$$H^i_{R_+}(K_R)_{(c,e)} = H^i_{\mathcal{M}_2}(K_R)_{(c,e)},$$

and so *(ii)* and *(iii)* are equivalent. To get the vanishing of the local cohomology modules with respect to the maximal ideal $\mathcal{M}$, it is just enough to take $c > \max\{a^1_*(R), a^1_*(K_R)\}$. By Theorem 3.4.6 there exists $\alpha > 0$ such that $a_*(I^e) < de + \alpha$, $a_*(K^e) < de + \alpha$, so by taking $c \geq de + \alpha$ we have $H^i_{\mathcal{M}_1}(R)_{(c,e)} = H^i_{\mathfrak{m}}(I^e)_c = 0$ and $H^i_{\mathcal{M}_1}(K_R)_{(c,e)} = H^i_{\mathfrak{m}}(K^e)_c = 0$. $\square$

Also note that the last proposition holds if we replace the condition for all $c \geq ef$ and $e > 0$ by the following one: for all $c \geq de + \alpha$ and $e > 0$. As a direct consequence, we obtain:

Corollary 3.5.7 *Assume that R has some Cohen-Macaulay diagonal. If $a^2_*(R) \leq 0$ and $a^2_*(K_R) \leq 0$, there exists α such that $k[(I^e)_c]$ is Cohen-Macaulay for all $c \geq de + \alpha$ and $e > 0$.*

Proof. It is a direct consequence of Proposition 3.5.6 by noting that for any i and $e > 0$ we have $H^i_{\mathcal{M}_2}(R)_{(c,e)} = 0$, $H^i_{\mathcal{M}_2}(K_R)_{(c,e)} = 0$ by Proposition 2.1.18. $\square$

Remark 3.5.8 The converse of the last corollary is not true. Let us take the homogeneous ideal $I = (x^7, y^7, x^6y + x^2y^5)$ in the polynomial ring $A = k[x, y]$. We have $\mathfrak{m}^{14} \subset I$, so $(I^e)_c = A_c$ for any $c \geq 14e$, and then $k[(I^e)_c] = k[x^a y^b \mid a + b = c]$ is Cohen-Macaulay for all these c, e. But $a^2_*(R) = 4 > 0$ by [HM, Example 3.13].

Chapter 4

Gorenstein coordinate rings of blow-up schemes

Let $Y = \mathrm{Proj}(A)$ be a closed subscheme of $\mathbb{P}_k^{n-1}$, and let X be the blow-up of Y along $\mathcal{I} = \widetilde{I}$, where I is a homogeneous ideal in A. If I is generated by forms of degree $\leq d$, we have already shown that for any $c \geq de + 1$ the ring $k[(I^e)_c]$ is the homogeneous coordinate ring of a projective embedding of X in $\mathbb{P}_k^{N-1}$, where $N = \dim_k (I^e)_c$. In this chapter we are interested in the (quasi-) Gorenstein property of the rings $k[(I^e)_c]$. The results work in the case of the blow-up of the projective space $\mathbb{P}_k^{n-1}$.

If the Rees algebra is Cohen-Macaulay and the associated graded ring is Gorenstein we will determine exactly for which pairs (c, e) the ring $k[(I^e)_c]$ is quasi-Gorenstein and, in particular, we will obtain that there is just a finite set of diagonals with this property. This result can be applied to several families of ideals. In particular, to any complete intersection ideal, extending in this way [CHTV, Corollary 4.7], and to the ideal generated by the maximal minors of a generic matrix.

After that, we show that there are always at most a finite number of rings $k[(I^e)_c]$ which are quasi-Gorenstein and we give upper bounds for such diagonals whenever $R_A(I)$ is Cohen-Macaulay.

Finally, we prove that under some restrictions the existence of a diagonal (c, e) such that $k[(I^e)_c]$ is quasi-Gorenstein forces the associated graded ring to be Gorenstein.

At the end of the chapter we apply our results to Room surfaces. These surfaces are obtained by blowing-up $\mathbb{P}^2_k$ along $\binom{d+1}{2}$ points, $d \geq 2$, which do not lie in any curve of degree $d-1$ and then embedding in $\mathbb{P}^{2d+2}_k$. We will show that the only Room surface which is Gorenstein is the del Pezzo sestic surface in $\mathbb{P}^6$, so recovering that well known result (see [GG, Example 1]).

4.1 The case of ideals whose form ring is Gorenstein

Throughout all the chapter we shall use the following notations. $A = k[X_1, ..., X_n]$ will denote the usual polynomial ring with coefficients in a field k, and $I \subset A$ a homogeneous ideal minimally generated by forms $f_1, ..., f_r$ of degrees $d_1 \leq ... \leq d_r = d$. We put $u = \sum_{j=1}^{r} d_j$. Let $S = k[X_1, ..., X_n, Y_1, ..., Y_r]$ be the polynomial ring with the grading obtained by setting $\deg X_i = (1, 0)$ for $i = 1, ..., n$, $\deg Y_j = (d_j, 1)$ for $j = 1, ..., r$; so that $R = R_A(I)$ and $G = G_A(I)$ can be seen in a natural way as bigraded S-modules. We will assume $n \geq r \geq 2$.

Notice that any diagonal S_Δ of the polynomial ring S is Cohen-Macaulay by Corollary 2.1.8. We begin this section by showing that, on the contrary, S_Δ is Gorenstein only for a finite number of diagonals. Furthermore, we may determine them.

Proposition 4.1.1 *S_Δ is Gorenstein if and only if $\frac{r}{e} = \frac{n+u}{c} = l \in \mathbb{Z}$. Then $a(S_\Delta) = -l$.*

Proof. Let $T = S_\Delta = \bigoplus_{s \geq 0} U_s$, where U_s is the k-vector space generated by the monomials $X_1^{\alpha_1} \cdots X_n^{\alpha_n} Y_1^{\beta_1} \cdots Y_r^{\beta_r}$ with $\alpha_i, \beta_j \geq 0$ satisfying the equations

$$(\star) \quad \begin{cases} \sum_{i=1}^{n} \alpha_i + \sum_{j=1}^{r} d_j \beta_j = cs \\ \\ \sum_{j=1}^{r} \beta_j = es \end{cases}$$

By Corollary 2.1.4, we have

$$H^{n+r-1}_m(T) \cong H^{n+r}_{\mathcal{M}}(S)_\Delta \cong (\bigoplus_{\alpha<0, \beta<0} kX^\alpha Y^\beta)_\Delta.$$

Therefore, $K_T = \underline{\mathrm{Hom}}_k(H^{n+r-1}_m(T), k) = \bigoplus_{s \geq 1} V_s$ with V_s the k-vector space generated by the monomials $X_1^{\alpha_1} \cdots X_n^{\alpha_n} Y_1^{\beta_1} \cdots Y_r^{\beta_r}$, and $\alpha_i > 0, \beta_j > 0$ which

satisfy $(\star)$. Since T is Cohen-Macaulay, T is Gorenstein if and only if $K_T \cong T(a(T))$.

Assume first that $\frac{r}{e} = \frac{n+u}{c} = l \in \mathbb{Z}$. Then, the multiplication by $X_1 \cdots X_n Y_1 \cdots Y_r \in T_l$ induces an isomorphism $T \cong K_T(l)$ and so T is Gorenstein with $a(T) = -l$.

To prove the converse set $(\alpha, \beta) = (\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_r)$ with $\alpha_i, \beta_j > 0$ and assume the contrary. This means that $(\mathbf{1}, \mathbf{1})$ is not a solution of $(\star)$ for any s. On the other hand, the set of solutions of $(\star)$ for some s is partially ordered by means of $(\alpha, \beta) \leq (\gamma, \rho) \Longleftrightarrow \alpha_i \leq \gamma_i,\ \beta_j \leq \rho_j,\ \forall i, j$. Then, one can easily check that for any $i = 1, \dots, n,\ j = 1, \dots, r$ there exists a solution of $(\star)$ for some s such that $\alpha_i = \beta_j = 1$. This implies the existence of at least two minimal solutions, and so T is not Gorenstein. $\square$

Remark 4.1.2 Note that the number of minimal elements in the set of solutions of the system $(\star)$ coincides with the type of S_Δ. It is not difficult to see that if S_Δ is not Gorenstein, then its type is $\geq r$.

Remark 4.1.3 Throughout the chapter we assume $r \geq 2$. In the case that $r = 1$ we have that I is a principal ideal, and $R_A(I) \cong S = k[X_1, \dots, X_n, Y]$. Then, it is easy to check that S_Δ is always Cohen-Macaulay, and S_Δ is Gorenstein if and only if $\Delta = (n + d, 1)$.

The last proposition leads to the question of whether there exist diagonals (c, e) such that $k[(I^e)_c]$ be quasi-Gorenstein, and how we can determine them. Our answer will be partially based on the following proposition which links the diagonal of the canonical module of $R_A(I)$ to the canonical module of the diagonal of $R_A(I)$. It is stated and proved for complete intersection ideals in [CHTV, Proposition 4.5], but in fact the same statement and proof are valid in general. We include the proof for completeness.

Proposition 4.1.4 $K_{R_\Delta} = (K_R)_\Delta$.

Proof. Let us consider a presentation of R as S-module

$$0 \to C \to S \to R \to 0,$$

which leads to the bigraded exact sequence of local cohomology modules

$$0 \to H^{n+1}_{\mathcal{M}}(R) \to H^{n+2}_{\mathcal{M}}(C) \to H^{n+2}_{\mathcal{M}}(S) \to 0.$$

Similarly, we get the graded exact sequence

$$0 \to H^n_m(R_\Delta) \to H^{n+1}_m(C_\Delta) \to H^{n+1}_m(S_\Delta) \to 0.$$

On the other hand, by Corollary 2.1.4 we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & H^{n+1}_{\mathcal{M}}(R)_\Delta & \to & H^{n+2}_{\mathcal{M}}(C)_\Delta & \to & H^{n+2}_{\mathcal{M}}(S)_\Delta & \to & 0 \\ & & \varphi^n_R \uparrow & & \varphi^{n+1}_C \uparrow & & \varphi^{n+1}_S \uparrow & & \\ 0 & \to & H^n_m(R_\Delta) & \to & H^{n+1}_m(C_\Delta) & \to & H^{n+1}_m(S_\Delta) & \to & 0 \end{array}$$

where φ^{n+1}_C, φ^{n+1}_S are isomorphisms, and so φ^n_R is also an isomorphism. Therefore $H^n_m(R_\Delta) \cong H^{n+1}_{\mathcal{M}}(R)_\Delta$ and we get

$$K_{R_\Delta} = \underline{\mathrm{Hom}}_k(H^n_m(R_\Delta), k) = \underline{\mathrm{Hom}}_k(H^{n+1}_{\mathcal{M}}(R)_\Delta, k) =$$

$$= \underline{\mathrm{Hom}}_k(H^{n+1}_{\mathcal{M}}(R), k)_\Delta = (K_R)_\Delta. \ \square$$

Remark 4.1.5 The hypothesis $n \geq r \geq 2$ fixed before is only used in this chapter to prove Proposition 4.1.4, and of course its applications. Nevertheless, the isomorphism $K_{R_\Delta} = (K_R)_\Delta$ is also valid if $n, r \geq 2$, I is equigenerated and R is Cohen-Macaulay. To prove this, assume $r > n$ (otherwise we may apply Proposition 4.1.4). Let us consider the bigraded minimal free resolution of R over S

$$0 \to D_{r-1} \to \ldots \to D_1 \to D_0 = S \to R \to 0,$$

with $D_p = \bigoplus_{(a,b)\in\Omega_p} S(a,b)$. Since R is Cohen-Macaulay, we have $-b \leq r-1$ and $-a \geq -bd$ for all $(a,b) \in \Omega_R$ by Lemma 3.4.10. On the other hand, recall from Corollary 2.1.7 that $H^r_{\mathcal{M}_2}(S(a,b))_{(cs,es)} \neq 0$ if and only if $\frac{bd-a}{c-ed} \leq s \leq \frac{-b-r}{e}$, so we get $H^r_{\mathcal{M}_2}(D_p)_\Delta = 0$ for all p. Since $r > n$, we also have that $H^i_{\mathcal{M}_1}(D_p)_\Delta = H^i_{\mathcal{M}_2}(D_p)_\Delta = 0$ for all $i > n$. Then, by Proposition 2.1.10 and Proposition 2.1.3 we have that φ^{n+1}_C is an isomorphism, and the same proof as in Proposition 4.1.4 shows that $K_{R_\Delta} = (K_R)_\Delta$.

This means that all the results we are going to prove in this chapter are also valid if $n, r \geq 2$, I is equigenerated and $R_A(I)$ is Cohen-Macaulay.

In view of Proposition 4.1.4 any information on the bigraded structure of K_R will be of interest. Let B be a d-dimensional local ring, $d \geq 1$, which has a canonical module K_B and $I \subset B$ an ideal of positive height such that $R_B(I)$ is Cohen-Macaulay. In [TVZ, Theorem 2.2] it is given a description of $K_{R_B(I)}$

in terms of a filtration of submodules of K_B. Assume now that $B = \bigoplus_{n\geq 0} B_n$ is a positively graded ring of positive dimension over a local ring B_0, which has a canonical module K_B. Let $I \subset B$ be a homogeneous ideal of positive height. Then, the Rees algebra $R_B(I)$ has a bigraded structure by means of $[R_B(I)]_{(i,j)} = (I^j)_i t^j$ for all $i, j \geq 0$. We also have a bigraded structure on the form ring by means of $[G_B(I)]_{(i,j)} = (I^j)_i/(I^{j+1})_i$ for all $i, j \geq 0$.

Then, the proof of [TVZ, Theorem 2.2] may be "bigraded" and we thus obtain a description of the bigraded structure of $K_{R_B(I)}$. Namely, we get:

Theorem 4.1.6 *With the notation above assume that $R_B(I)$ is Cohen-Macaulay. Then, there exists a homogeneous filtration $\{K_m\}_{m\geq 0}$ of K_B and isomorphisms of bigraded modules such that*

$$K_{R_B(I)} \cong \bigoplus_{(l,m),\, m\geq 1} [K_m]_l\,,$$

$$K_{G_B(I)} \cong \bigoplus_{(l,m),\, m\geq 1} [K_{m-1}]_l/[K_m]_l.$$

Several other results of [TVZ] may also be "bigraded". In particular [TVZ, Lemma 4.1] which makes precise when the canonical module of the Rees algebra has the expected form. Recall that $K_{R_B(I)}$ has the expected form if

$$K_{R_B(I)} \cong Bt \oplus Bt^2 \oplus \cdots \oplus Bt^l \oplus It^{l+1} \oplus I^2 t^{l+2} \oplus \cdots\,,$$

for some $l \geq 0$. This definition was introduced by J. Herzog, A. Simis and W. Vasconcelos in [HSV2]. We still use the same notation and again omit the proof.

Corollary 4.1.7 *Assume $R_B(I)$ is Cohen-Macaulay and $G_B(I)$ is quasi-Gorenstein. Set $\mathbf{a}(G_B(I)) = (-b, -a)$. Then $K_B \cong B(-b)$ and*

$$K_{R_B(I)} \cong \bigoplus_{(l,m),\, m\geq 1} [I^{m-a+1}]_{l-b},$$

where $I^n = B$ if $n \leq 0$.

Note that $-a$ coincides with the usual a-invariant of $G_B(I)$. By Ikeda-Trung's criterion [IT] it is always negative if $R_B(I)$ is Cohen-Macaulay, and it has been calculated in many cases (see for instance [HRZ], [GH]). As for

b, it is clear that under the hypothesis of Corollary 4.1.7 we get $-b = a(B)$. It is then also easy to compute the bigraded a-invariant of $R_B(I)$. Namely, we get that if $a = 1$ then $\mathbf{a}(R_B(I)) = (-d_1 + a(B), -1)$, and if $a > 1$ then $\mathbf{a}(R_B(I)) = (a(B), -1)$.

Remark 4.1.8 Assume that $B = A = k[X_1, \ldots, X_n]$ and I is a complete intersection ideal. Then, the Eagon-Northcott complex provides a $\mathbb{Z}^2$-graded minimal free resolution of $R_A(I)$. Following the proof of Yoshino [Yos] it is possible to see that

$$K_{R_A(I)} = J((r-2)d_1 - n, -1)$$

with $J = (f_1^{r-2}, f_1^{r-2}\, t, \ldots, f_1^{r-2}\, t^{r-2})R_A(I)$.

Observe that in this case $a(G_A(I)) = (-n, -r)$ and by Corollary 4.1.7

$$K_{R_A(I)} = \bigoplus_{(l,m),\, m\geq 1} [I^{m-r+1}]_{l-n}.$$

A straightforward computation shows that, in fact, multiplication by f_1^{r-2} provides an explicit isomorphism

$$\bigoplus_{(l,m),\, m\geq 1} [I^{m-r+1}]_{l-n} \cong J((r-2)d_1 - n, -1).$$

Let us now assume that $I \subset A = k[X_1, \ldots, X_n]$ is a homogeneous ideal whose form ring is Gorenstein. We are now ready to prove the main result of this section determining the quasi-Gorenstein diagonals of $R_A(I)$. Namely,

Theorem 4.1.9 *Assume* $\mathrm{ht}\,(I) \geq 2$, $\dim(A/I) > 0$, *and* $G_A(I)$ *is Gorenstein. Set* $a = -a^2(G_A(I))$. *Then* $k[(I^e)_c]$ *is quasi-Gorenstein if and only if* $\frac{n}{c} = \frac{a-1}{e} = l_0 \in \mathbb{Z}$. *In this case,* $a(k[(I^e)_c]) = -l_0$.

Proof. Note that the Rees algebra R is Cohen-Macaulay by using a result of Lipman [Li, Theorem 5]. Now, by applying Corollary 4.1.7 we have that $b = -a(A) = n$ and $K_R = \bigoplus_{(l,m), m\geq 1}[I^{m-a+1}]_{l-n}$, so that by Proposition 4.1.4 we get $K_{R_\Delta} = (K_R)_\Delta = \bigoplus_{l\geq 1}[I^{el-a+1}]_{cl-n}$. Let $l_0 = \min\{l \in \mathbb{Z} \mid l \geq \frac{n}{c}\}$, $s = a - 1 - el_0$. We shall now distinguish three cases.

If $s = 0$, then the first non-zero component of K_{R_Δ} is $[K_{R_\Delta}]_{l_0} = [I^{el_0-a+1}]_{cl_0-n} = A_{cl_0-n}$, so that if R_Δ is quasi-Gorenstein $cl_0 - n = 0$ and we get that $l_0 = \frac{n}{c} = \frac{a-1}{e}$ and $a(R_\Delta) = -l_0$. Conversely, if $l_0 = \frac{n}{c} = \frac{a-1}{e}$ then

$[K_{R_\Delta}]_{l_0+m} = [I^{el_0-a+1+em}]_{cl_0+cm-n} = [I^{em}]_{cm} = [R_\Delta]_m$ for all m and so R_Δ is quasi-Gorenstein.

If $s<0$, let $l_1 = \min\{l \mid el-a+1>0, cl-n \geq d_1(el-a+1)\}$. Then $l_1 \geq l_0$ and the first non-zero component of K_{R_Δ} is $[K_{R_\Delta}]_{l_1} = [I^{el_1-a+1}]_{cl_1-n}$. In particular, $a(R_\Delta) = -l_1$. Assume R_Δ is quasi-Gorenstein. Then $K_{R_\Delta} \cong R_\Delta(-l_1)$ and so $[K_{R_\Delta}]_{l_1} \cong k$. This implies that $cl_1 - n = d_1(el_1 - a + 1)$: If $cl_1 - n - d_1(el_1 - a+1) = r > 0$ we may choose two lineary independent forms $g, h \in A_r$ such that $gf_1^{el_1-a+1}, hf_1^{el_1-a+1} \in [I^{el_1-a+1}]_{cl_1-n} \cong k$, which is a contradiction. From the isomorphism one gets that K_{R_Δ} is generated by $f_1^{el_1-a+1}$ as R_Δ-module. Now let $f_j \notin \mathrm{rad}\,(f_1)$ (it exists because $\mathrm{ht}\,(I) \geq 2$), and choose m such that $m(c-d_je) > d_j - d_1$ and there exists $f \in A_{d_1+cm-d_j(em+1)}$ such that $(f, f_1) = 1$. Then $f_1^{el_1-a} f_j^{em+1} f \in [I^{el_1-a+1+em}]_{d_1(el_1-a+1)+cm} = f_1^{el_1-a+1}[I^{em}]_{cm}$, and we get $f_j^{em+1} f \in (f_1)$ which is a contradiction.

If $s > 0$, the first non-zero component of K_{R_Δ} is $[I^{el_0-a+1}]_{cl_0-n} = A_{cl_0-n}$, so if R_Δ is quasi-Gorenstein we get $cl_0 - n = 0$. Furthermore, for all $m \geq 1$ we have $[K_{R_\Delta}]_{l_0+m} = [I^{-s+em}]_{cl_0-n+cm} = [I^{-s+em}]_{cm} \cong [I^{em}]_{cm}$. Since $s > 0$ and $[I^{em}]_{cm} \subset [I^{-s+em}]_{cm}$ this isomorphism is possible if and only if $[I^{em}]_{cm} = [I^{-s+em}]_{cm}$. Now choose X_i such that $X_i \notin \mathrm{rad}\,(I)$ (it always exists because $\dim(A/I) > 0$) and m with $em - s \geq 1$. For any j consider $F_j = X_i^{\alpha_j} f_j^{em-s}$ where $\alpha_j = cm - d_j(em-s) \geq 1$, and assume $[I^{em}]_{cm} = [I^{-s+em}]_{cm}$. Then $F_j \in [I^{em-s}]_{cm}$ and so $X_i^{\alpha_j} f_j^{em-s} \in I^{em}$. Now let $f_1^{c_1} \ldots f_r^{c_r}$ such that $c_1 + \ldots + c_r \geq r(em-s)$. This implies that there exists l with $c_l \geq em - s$ and so $X_i^{\alpha_1} f_1^{c_1} \ldots f_r^{c_r} = X_i^{\alpha_1} f_l^{em-s} f_1^{c_1} \ldots f_l^{c_l - em+s} \ldots f_r^{c_r} \in I^{c_1+\ldots+c_r+s}$, since $\alpha_1 \geq \alpha_i$ for all i. Thus we get $X_i^{\alpha} I^h \subset I^{h+s}$ for $h >> 0$, which implies that $X_i^\alpha \in I^s \subset I$ since $R_A(I)$ is Cohen-Macaulay (see [TVZ, Lemma 4.3]). But this contradicts $X_i \notin \mathrm{rad}\,(I)$ and so R_Δ is not quasi-Gorenstein. $\Box$

The remaining cases $\mathrm{ht}\,(I) = 1, n$ in the above theorem are studied separately in the following remarks.

Remark 4.1.10 If $\mathrm{ht}\,(I) = 1$ then $k[(I^e)_c]$ is never quasi-Gorenstein. In fact, by [TVZ, Proposition 4.6], $a^2(G_A(I)) = -1$ and so $a = 1$. Following the same proof as in Theorem 4.1.9 we have that $s = -el_0 < 0$. On the other hand, since $\mathrm{ht}\,(I) = 1$ we may write $I = gJ$, with $\mathrm{ht}\,(J) \geq 2$, $J = (\overline{f}_1, \ldots, \overline{f}_r)$ and $f_j = \overline{f}_j g$ for all j. The same argument as in Theorem 4.1.9 for the case $s < 0$ but taking $\overline{f}_j \notin \mathrm{rad}\,(\overline{f}_1)$ and $f \in A_{d_1+cm-d_j(em+1)}$ such that $(f, \overline{f}_1) = 1$ leads to $\overline{f}_j^{em+1} f \in (\overline{f}_1)$, which is a contradiction.

Remark 4.1.11 If $\dim(A/I) = 0$, then the condition $\frac{n}{c} = \frac{a-1}{e} = l_0 \in \mathbb{Z}$ is sufficient but not necessary for $k[(I^e)_c]$ to be quasi-Gorenstein. For instance, let $A = k[X_1, X_2, X_3]$ and $I = (X_1, X_2, X_3)$. Note that $n = 3 \geq r = 3 \geq 2$, G is Gorenstein and $a = 3$. Then, by Corollary 4.1.7 we have that $K_R = \bigoplus_{(l,m),m\geq 1}[I^{m-2}]_{l-3}$. According to Proposition 4.1.4, by taking the $(3,1)$-diagonal we have

$$K_{R_\Delta} = \bigoplus_{l\geq 1}[I^{l-2}]_{3(l-1)} = \bigoplus_{l\geq 1} A_{3(l-1)} = (\bigoplus_{l\geq 0} A_{3l})(-1) = R_\Delta(-1),$$

and so $R_\Delta = k[I_3]$ is quasi-Gorenstein. In this case, $\frac{n}{c} = 1 \neq 2 = \frac{a-1}{e}$.

As a consequence of Theorem 4.1.9 we obtain the following result for the case of complete intersection ideals. It generalizes [CHTV, Corollary 4.7] where the case of ideals generated by two elements was considered.

Corollary 4.1.12 *Let $I \subset k[X_1, \ldots, X_n]$ be a complete intersection ideal minimally generated by r forms of degrees $d_1 \leq \ldots \leq d_r = d$, with $r < n$. Then for $c \geq de + 1$, $k[(I^e)_c]$ is Gorenstein if and only if $\frac{n}{c} = \frac{r-1}{e} = l_0 \in \mathbb{Z}$. In this case, $a(k[(I^e)_c]) = -l_0$.*

Proof. Since $a^2(G_A(I)) = -r$, by Theorem 4.1.9 we have that $k[(I^e)_c]$ is quasi-Gorenstein if and only if $\frac{n}{c} = \frac{r-1}{e} = l_0 \in \mathbb{Z}$. But then $u + (e-1)d - n \leq rd + ed - d - n = (r-1)d + de - n = e\frac{n}{c}d + de - n = n(\frac{ed-c}{c}) + de \leq de < c$, and according to Proposition 3.4.5, $k[(I^e)_c]$ is also Cohen-Macaulay and so Gorenstein. □

We may also study the ideals generated by the maximal minors of a generic matrix. We thank A. Conca for suggesting to consider this case.

Example 4.1.13 Let $\mathbf{X} = (X_{ij})$ be a generic matrix, with $1 \leq i \leq n$, $1 \leq j \leq m$ and $m \leq n$. Let us consider $I \subset A = k[\mathbf{X}]$ the ideal generated by the maximal minors of $\mathbf{X}$, where k is a field. In this case, the Rees algebra $R_A(I)$ is Cohen-Macaulay and the form ring $G_A(I)$ is Gorenstein [EH, Theorem 3.5]. Moreover, it has been proved by A. Conca (personal communication) that all the diagonals of $R_A(I)$ are Cohen-Macaulay (see also Example 5.2.23).

Now we want to study the Gorenstein property of these diagonals. Note that I is an equigenerated ideal whose Rees algebra is Cohen-Macaulay, so we can apply Theorem 4.1.9 thanks to Remark 4.1.5. Since I is generically a

complete intersection, we have that $a^2(G_A(I)) = -\mathrm{ht}\,(I) = -(n-m+1)$. We shall distinguish two cases.

If $m < n$, then $\mathrm{ht}\,(I) \geq 2$, and we get that $k[(I^e)_c]$ is Gorenstein if and only if $\frac{nm}{c} = \frac{n-m}{e} \in \mathbb{Z}$. So there exists always at least one diagonal which is Gorenstein by taking $c = nm, e = n-m$.

If $m = n$, note that I is a principal ideal and so the Rees algebra is isomorphic to a polynomial ring. Then the only diagonal which is Gorenstein occurs when $c = n(n+1), e = 1$ by Remark 4.1.3.

4.2 Restrictions to the existence of Gorenstein diagonals. Applications.

In Section 4.1 we have proved that under the assumptions of Theorem 4.1.9 there is just a finite set of diagonals (c,e) such that $k[(I^e)_c]$ is quasi-Gorenstein. Our next result shows that this holds in general.

Proposition 4.2.1 *There exist at most a finite number of diagonals* (c,e) *such that* $k[(I^e)_c]$ *is quasi-Gorenstein.*

Proof. Let $w_1, ..., w_m \in K_R$ be a homogeneous system of generators of K_R as R-module with $\deg w_i = (\alpha_i, \beta_i)$ for all i, and so $K_R = \sum_{i=1}^m Rw_i$. Note that since R is a domain K_R is torsion free. For any diagonal $\Delta = (c,e)$ we then have by Proposition 4.1.4 that for all $l \geq 1$

$$[K_{R_\Delta}]_l = \sum_{i=1,...,m,\, el-\beta_i \geq 0} [I^{el-\beta_i}]_{cl-\alpha_i} w_i.$$

If R_Δ is quasi-Gorenstein there exists an integer l such that $[K_{R_\Delta}]_l \cong k$ and so $[I^{el-\beta_i}]_{cl-\alpha_i} \neq 0$ for some i $(\star)$. We shall distinguish two cases.

Assume first that I is an equigenerated ideal in degree d. Then condition $(\star)$ implies that $el-\beta_i = 0$ and $cl-\alpha_i \geq 0$ or $el-\beta_i > 0$ and $cl-\alpha_i \geq d(el-\beta_i)$. If $el-\beta_i = 0$, then $k \cong [K_{R_\Delta}]_l \supset A_{cl-\alpha_i} w_i$ and since K_R is torsion-free we get $cl - \alpha_i = 0$. Hence (c,e) satisfies $\frac{\beta_i}{e} = \frac{\alpha_i}{c} = l \in \mathbb{Z}$ and the statement holds. If $el - \beta_i > 0$ then $k \cong [K_{R_\Delta}]_l \supset [I^{el-\beta_i}]_{cl-\alpha_i} w_i$ which is impossible since K_R is torsion free and $cl - \alpha_i \geq d(el-\beta_i)$.

Assume now that I is not equigenerated. Condition $(\star)$ implies that $el - \beta_i = 0$ and $cl - \alpha_i \geq 0$ or $el - \beta_i > 0$ and $cl - \alpha_i \geq d_1(el - \beta_i)$. In the

first case we may proceed as before to get the statement. In the second case we have that $k \cong [K_{R_\Delta}]_l \supset [I^{el-\beta_i}]_{cl-\alpha_i} w_i$ and so $cl - \alpha_i = d_1(el - \beta_i)$ and $d_1 < d_2$. Then $\alpha_i - d_1\beta_i = cl - d_1 el \geq c - d_1 e \geq (d - d_1)e$ since $l \geq 1$ and $c \geq de + 1 > de$. Thus we obtain the inequality $e \leq \frac{\alpha_i - d_1\beta_i}{d - d_1}$ and for each e, we have $c \leq d_1 e + \alpha_i - d_1\beta_i$. In any case, these inequalties hold for at most a finite number of diagonals and so we get the result. $\square$

For a real number x, let us denote by $\lceil x \rceil = \min\{m \in \mathbb{Z} \mid m \geq x\}$. If the Rees algebra $R_A(I)$ is Cohen-Macaulay we can also give bounds for the diagonals (c,e) such that $k[(I^e)_c]$ is quasi-Gorenstein.

Proposition 4.2.2 *Assume that* $\mathrm{ht}\,(I) \geq 2$ *and* $R_A(I)$ *is Cohen-Macaulay. Let* $a = -a^2(G_A(I))$. *If* $k[(I^e)_c]$ *is quasi-Gorenstein, then* $e \leq a - 1$ *and* $c \leq n$. *Moreover, if* $\dim(A/I) > 0$ *then* $\lceil \frac{a}{e} \rceil - 1 = \frac{n}{c} \in \mathbb{Z}$. *In particular, if* $a = 1$ *there are no diagonals* (c,e) *such that* $k[(I^e)_c]$ *is quasi-Gorenstein.*

Proof. By Theorem 4.1.6, there exists a homogeneous filtration $\{K_m\}_{m\geq 0}$ of K_A such that $K_R \cong \bigoplus_{m\geq 1} K_m$ and $K_G \cong \bigoplus_{m\geq 1} K_{m-1}/K_m$. Bigrading the proof of [TVZ, Corollary 2.5], we have that $K_m = \mathrm{Hom}_A(I, K_{m+1})$ for every $m \geq 0$. Note that K_A may be viewed as an ideal of A. Assume that R_Δ is quasi-Gorenstein. Then there is an integer l_0 such that $[K_{R_\Delta}]_{l_0} \cong k$. By Proposition 4.1.4 we may find an element $f \in [K_{el_0}]_{cl_0} = [K_R]_{(cl_0, el_0)}$, $f \neq 0$, $K_{R_\Delta} = R_\Delta f$.
CLAIM: $K_{el_0} = Af$.

To prove the claim we first show that for any $g \in K_{el_0}$, $g \neq 0$, then g has degree $\geq cl_0$. Assume the contrary: $\deg\, g = k < cl_0$. Then $[Ag]_{cl_0} = A_{cl_0 - k} g \subset [K_{el_0}]_{cl_0} \cong k$. But since $cl_0 - k > 0$, $\dim_k A_{cl_0-k} > 1$, so we get a contradiction.

Now let $g \in K_{el_0}$. If $\deg g = cl_0$, then $g \in Af$ because $[K_{el_0}]_{cl_0} = kf$. Let us assume that $\deg\, g = k > cl_0$. Then, for each $l > 0$, $[I^{el}]_{cl} f + [I^{el}]_{c(l_0+l)-k}\, g \subset [K_{e(l_0+l)}]_{c(l_0+l)} \cong [I^{el}]_{cl}$ as k-vector spaces, and so $[I^{el}]_{c(l_0+l)-k}\, g \subset [I^{el}]_{cl} f$. Now let $I^{el} = (F_1, \ldots, F_t)$ where F_i is a homogeneous polynomial of degree $\leq del$ for all i, and set $\alpha = c(l_0 + l) - k - \deg F_i$. Note that for $l >> 0$, $\alpha \geq c(l_0 + l) - k - del = (c - de)l + cl_0 - k > 0$ and we can find $h \in A_\alpha$ such that $(h, f) = 1$. Then $hgF_i \in [I^{el}]_{c(l_0+l)-k}\, g \subset [I^{el}]_{cl} f \subset Af$ and we have that $gF_i \in Af$ for all i. Thus $I^{el} g \subset (f)$ and writing $g = d\overline{g}$, $f = d\overline{f}$ with $(\overline{f}, \overline{g}) = 1$ we get $I^{el}\overline{g} \subset A\overline{f}$. If $g \notin Af$, then $\overline{f} \notin k$ and so $I^{el} \subset (\overline{f})$ which is absurd because $\mathrm{ht}\,(I) \geq 2$.

Now, as $\mathrm{grade}(I) \geq 2$ we have $K_m = K_{el_0}$ for all $m \leq el_0$, which implies that $K_A = K_{el_0}$ and so $c \leq cl_0 = n$. Furthermore, $e \leq el_0 \leq \min\{m \mid K_m \subsetneq K_{m-1}\} - 1 = a - 1$.

Finally assume that $\dim(A/I) > 0$. We shall distinguish two cases. If $e = 1$ we have that $K_{l_0+1} \subsetneq K_{l_0}$: If not, then $I_c \cong [K_{l_0+1}]_{c(l_0+1)} = [Af]_{c(l_0+1)} \cong A_c$ which is absurd if $\dim(A/I) > 0$. Therefore $a = l_0 + 1 = \frac{n}{c} + 1$. If $e > 1$, let $\widetilde{\Delta} = (c, 1)$ and $\widetilde{R} = R_A(I^e)$. Note that $\widetilde{R}_{\widetilde{\Delta}} = R_\Delta$ is quasi-Gorenstein. Applying the case before we obtain that $-a(G_A(I^e)) = \frac{n}{c} + 1$. By [HRZ, Proposition 2.6], $a(G_A(I^e)) = [\frac{-a}{e}] = -\lceil \frac{a}{e} \rceil$ and so $\lceil \frac{a}{e} \rceil - 1 = \frac{n}{c} = l_0 \in \mathbb{Z}$. $\square$

Our next result shows that in some cases the existence of a diagonal (c, e) such that $k[(I^e)_c]$ is quasi-Gorenstein forces the form ring to be Gorenstein. It may be seen as a converse of Theorem 4.1.9 for those cases.

Theorem 4.2.3 *Assume that $R_A(I)$ is Cohen-Macaulay,* $\mathrm{ht}\,(I) \geq 2$*, $l(I) < n$ and I is equigenerated. If there exists a diagonal (c, e) such that $k[(I^e)_c]$ is quasi-Gorenstein then $G_A(I)$ is Gorenstein.*

Proof. Let $\Delta = (c, e)$. Assume first that $e = 1$. We have seen in the proof of Proposition 4.2.2 that there exists a homogeneous filtration $\{K_m\}_{m \geq 0}$ of K_A such that $K_R \cong \bigoplus_{m \geq 1} K_m$ and $K_G \cong \bigoplus_{m \geq 1} K_{m-1}/K_m$, and an integer $l_0 = -a(R_\Delta)$ such that $K_0 = \ldots = K_{l_0} = Af$, with $f \in K_R$ and $\deg\, f = cl_0$. It is then clear that for all $m \geq 0$, $I^m f \subset K_{l_0+m}$ and so $[I^m]_{cm} f \subset [K_{l_0+m}]_{c(l_0+m)} \cong [I^m]_{cm}$ since R_Δ is quasi-Gorenstein. This implies that $[K_{l_0+m}]_{c(l_0+m)} = [I^m]_{cm} f$.

We want to show that $K_{l_0+m} = I^m f$ for all $m \geq 0$. Suppose that there exists m_0 such that $I^{m_0} f \subsetneq K_{l_0+m_0}$. Then let $g \in K_{l_0+m_0}$, $g \notin I^{m_0} f$ be a homogeneous element of degree k. Note that from the inclusion $K_{l_0+m_0} \subset K_{l_0} = Af$ we also have $g = f\overline{g}$ with $\overline{g} \notin I^{m_0}$.

If $k \geq c(l_0 + m_0)$ then for any $m > m_0$ we have $I^m f + I^{m-m_0} g \subset K_{l_0+m}$ and so $[I^m]_{cm} f + [I^{m-m_0}]_{c(l_0+m)-k}\, g \subset [K_{l_0+m}]_{c(l_0+m)} \cong [I^m]_{cm}$. Hence $[I^{m-m_0}]_{c(l_0+m)-k}\, g \subset [I^m]_{cm} f$ and we get that $[I^{m-m_0}]_{c(l_0+m)-k}\, \overline{g} \subset [I^m]_{cm}$. Let $\lambda = c(l_0 + m) - k - d(m - m_0) = (c - d)m + cl_0 + dm_0 - k$. For $m >> 0$ we have that $\lambda > 0$. Then, if $A_\lambda \overline{g} \subset I^{m_0}$ we would have that $\overline{g} \in (I^{m_0})^*$, the saturation of I^{m_0}. Since $G_A(I)$ is Cohen-Macaulay, we have that the inequality of Burch becomes an equality by [EH, Proposition 3.3], that is,

$$\inf_{j \geq 0}\{\mathrm{depth}(A/I^j)\} = \dim A - l(I).$$

Since $l(I) < n$, we then get $\operatorname{depth} A/I^{m_0} > 0$, and so $(I^{m_0})^* = I^{m_0}$. Hence $\overline{g} \in I^{m_0}$, which is a contradiction. We may conclude that there exist $\lambda > 0$, $h \in A_\lambda$ such that $\overline{g}h \notin I^{m_0}$. On the other hand, $\overline{g}h[I^{m-m_0}]_{d(m-m_0)} \subset \overline{g}[I^{m-m_0}]_{c(l_0+m)-k} \subset [I^m]_{cm}$. Since I is equigenerated we get $\overline{g}hI^{m-m_0} \subset I^m$. Therefore, $\overline{g}h \in (I^m : I^{m-m_0}) = I^{m_0}$ because R is Cohen-Macaulay. This is a contradiction.

If $k < c(l_0 + m_0)$, let us write $k = c(l_0 + m_0) - s$ with $s > 0$. Then $A_s g \subset [K_{l_0+m_0}]_{c(l_0+m_0)} = [I^{m_0}]_{cm_0} f$, and $\overline{g} \in (I^{m_0})^* = I^{m_0}$ which, as before, is a contradiction.

Hence we have proved that $K_{l_0+m} = I^m f$ for all $m \geq 0$, so

$$K_R = f(At \oplus \cdots \oplus At^{l_0} \oplus It^{l_0+1} \oplus \cdots),$$

i.e. K_R has the expected form. By [TVZ, Theorem 4.2] this implies that both $R_A(I^{l_0})$ and $G_A(I)$ are Gorenstein.

Finally assume $e > 1$, and denote by $\widetilde{\Delta} = (c, 1)$ and $\widetilde{R} = R_A(I^e)$. Then $\widetilde{R}_{\widetilde{\Delta}} = R_\Delta$ is quasi-Gorenstein and so there exists l_0 such that $R_A(I^{el_0})$ is Gorenstein. By [TVZ, Theorem 4.2] this implies again that $G_A(I)$ is Gorenstein. $\square$

Example 4.2.4 *(Room surfaces)* Let k be an algebraically closed field. Set $t = \binom{d+1}{2}$, with $d \geq 2$. Let $P_1, \ldots, P_t$ be a set of t distinct points in $\mathbb{P}^2_k$ which do not lie on a curve of degree $d - 1$. We assume further that there is not a subset of d points on a line if $d \geq 3$. We are going to study the rational projective surfaces which arise as embeddings of blowing-ups of $\mathbb{P}^2_k$ at this set of points via the linear system I_{d+1}.

Let I be the ideal defining the set of points $\{P_1, \ldots, P_t\}$. Since the points are not on a curve of degree $d - 1$, we have that I is generated by forms in degree d [GG], and so for any $c \geq d + 1$ the linear system I_c gives a projective embedding of the blow-up. For $c = d + 1$ the surface obtained is called Room surface.

Assume $d \geq 3$. Since there are not d points on a line, we also have that the rational map defined by the linear system I_d give an embedding of the blow-up in the projective space $\mathbb{P}^d_k$ (see [GG]), and the resulting surface is called White surface. A. Gimigliano proved that White surfaces have the defining ideal given by the 3×3 minors of a $3 \times d$ matrix of linear forms, and it has a linear minimal graded free resolution which comes from the Eagon-Northcott complex [Gi, Proposition 1.1]. By applying Theorem 1.3.4, we obtain $a(k[I_d]) = -1$ and

so by Lemma 2.3.7 the reduction number of I is $r(I) = a(k[I_d]) + l(I) = -1+3 = 2$. Moreover, the analytic deviation of I is $\mathrm{ad}(I) = l(I) - \mathrm{ht}\,(I) = 1$ and I is generically a complete intersection ideal. So according to [GN] we may conclude that $G_A(I)$ is Cohen-Macaulay and hence $a^2(G_A(I)) = r(I) - \mathrm{ht}\,(I) - 1 = -1$ by [GH, Proposition 2.4]. Therefore, $R_A(I)$ is also Cohen-Macaulay by using Ikeda-Trung's criterion. From Proposition 4.2.2 we get that there are not diagonals (c, e) such that $k[(I^e)_c]$ is Gorenstein, that is, there are not Gorenstein embeddings for the blow-up. In particular, $k[I_{d+1}]$ is not Gorenstein for $d \geq 3$.

If $d = 2$, by choosing the points to be [1:0:0], [0:1:0] and [0:0:1], we have $I = (X_1X_2, X_1X_3, X_2X_3)$. Notice that I has $\mu(I) = 3 = \mathrm{ht}\, I + 1$ and A/I is Cohen-Macaulay. Moreover, $\mu(I_\mathfrak{p}) \leq \mathrm{ht}\,(\mathfrak{p})$ for any prime ideal $\mathfrak{p} \supset I$. Then $G_A(I)$ is Gorenstein with and $a^2(G_A(I)) = -\mathrm{ht}\,(I) = -2$ by [HRZ]. Now, according to Theorem 4.1.9, $k[(I^e)_c]$ is quasi-Gorenstein if and only if $\frac{3}{c} = \frac{1}{e} \in \mathbb{Z}$. Hence $(3, 1)$ is the only diagonal with the quasi-Gorenstein property. This embedding corresponds to the del Pezzo sestic surface in $\mathbb{P}^6$.

With more generality, we may consider the blow-up of $\mathbb{P}^2_k$ at a set of t arbitrary distinct points.

Example 4.2.5 Let k be an algebraically closed field. Let $P_1, \dots, P_t$ be a set of t distinct points in $\mathbb{P}^2_k$, and let $I = \mathcal{P}_1 \cap \dots \cap \mathcal{P}_t$, where $\mathcal{P}_i \subset A = k[X_1, X_2, X_3]$ is the defining ideal of P_i. Now we consider the surfaces which arise as embeddings of the blow-up of $\mathbb{P}^2_k$ at these points via the linear systems $(I^e)_c$. We want to study the Gorenstein property of the rings $k[(I^e)_c]$.

Set $d = \mathrm{reg}(I)$. We will assume that $P_1, \dots, P_t$ do not lie on a curve of degree $d-1$ and that there is not a subset of d points on a line. Then, I is generated by forms in degree d and I_d defines a projective embedding of the blow-up [GG, Theorem A].

On the other hand, observe that $R_{A_\mathfrak{p}}(I_\mathfrak{p})$ is Cohen-Macaulay for all $\mathfrak{p} \in \mathrm{Proj}\,(A)$. Let $\mathcal{L} = \widetilde{I}\mathcal{O}_X$, $\mathcal{M} = \pi^*\mathcal{O}_{\mathbb{P}^2}(1)$, where $\pi : X \longrightarrow \mathbb{P}^2_k$ is the blow-up of $\mathbb{P}^2_k$ along $\mathcal{I}$. Then we have that $R^j\pi_*\mathcal{L}^e = 0$ for all $e \geq 0$, $j > 0$ and $\pi_*\mathcal{L}^e = \widetilde{I}^e$ for all $e \geq 0$ by Corollary 3.1.4. Therefore, for any $s \geq 0$, $\mathit{\Gamma}(X, \mathcal{L}^s \otimes \mathcal{M}^{sd}) = \mathit{\Gamma}(\mathbb{P}^2, \widetilde{I}^s(sd)) = (I^s)^*_{sd}$ and $H^i(X, \mathcal{L}^s \otimes \mathcal{M}^{sd}) = H^i(\mathbb{P}^2, \widetilde{I}^s(sd)) = H^{i+1}_\mathfrak{m}(I^s)_{sd}$ for $i \geq 1$. By [GGP, Theorem 1.1 and Corollary 1.4], we have $a_*(I^s) < \mathrm{reg}\,(I^s) \leq sd$ and $(I^s)^*_{sd} = (I^s)_{sd}$. Then, by Remark 3.1.1 we get $H^i_\mathfrak{m}(k[I_d])_s = 0$ for all $s \geq 0$. Furthermore, recall that the fiber cone F of I coincides with

$k[I_d]$ because I is generated in degree d, so we have $a_*(F) \leq -1$, and then $r(I) \leq \max_{i \leq 3}\{a_i(F)+i\} \leq 2$ by Lemma 2.3.7 . The analytic deviation of I is $\mathrm{ad}(I) = l(I) - \mathrm{ht}\,(I) = 1$ and I is generically a complete intersection ideal, so we may conclude by [GN] that $G_A(I)$ is Cohen-Macaulay and hence by [GH, Proposition 2.4]

$$a^2(G_A(I)) \leq r(I) - \mathrm{ht}\,(I) - 1 \leq -1.$$

So $R_A(I)$ is also Cohen-Macaulay by Ikeda-Trung's criterion.

If $r(I) = 2$ then $a^2(G_A(I)) = -1$ by [GH, Proposition 2.4]. Then, according to Proposition 4.2.2, there are not quasi-Gorenstein diagonals $k[(I^e)_c]$.

Otherwise, $r(I) \leq 1$. By [GN, Theorem 1.3], the case $r(I) = 1$ is not possible, so $r(I) = 0$ and then $G_A(I)$ is Gorenstein. Furthermore, $a^2(G_A(I)) = -\mathrm{ht}\,(I) = -2$ by [GH, Proposition 2.4]. Therefore, by Theorem 4.1.9, $k[(I^e)_c]$ is quasi-Gorenstein if and only if $\frac{3}{c} = \frac{1}{e} \in \mathbb{Z}$. So $k[I_3]$ is the only quasi-Gorenstein diagonal.

Remark 4.2.6 All throughout this chapter we have treated the case where $A = k[X_1, \ldots, X_n]$ is the polynomial ring and I is a homogeneous ideal in A satisfying $r \leq n$ or the assumptions in Remark 4.1.5. This set up was used to study the relationship between the canonical module of the Rees algebra R and the canonical modules of its diagonals R_Δ. Now let A be an arbitrary standard k-algebra and let I be a homogeneous ideal in A generated by r forms in degree $\leq d$. Set $\overline{n} = \dim A$. For any $c \geq de+1$, from the Mayer-Vietoris sequence (see Proposition 2.1.3) we have a graded monomorphism

$$\psi : (K_R)_\Delta \to K_{R_\Delta}$$

such that

(i) If $l(I) < \overline{n}$ or $r < \overline{n}$ or I equigenerated, then ψ_s is an isomorphism for any $s > 0$.

(ii) Assume $l(I) < \overline{n}$ or $a_2^*(R) < 0$. If $\mathrm{ht}\,(I) \geq 2$, $H^{\overline{n}}_{\mathfrak{m}}(A)_0 = 0$, $a(A) < c$, then ψ_s is an isomorphism for any $s \leq 0$.

(iii) Assume that R is Cohen-Macaulay. Then

$$\psi \text{ isomorphism } \iff \begin{cases} H^{\overline{n}}_{\mathfrak{m}}(A)_0 = 0 & \\ H^{\overline{n}}_{\mathfrak{m}}(I^{es})_{cs} = 0 & \text{for } s > 0 \\ H^{\overline{n}}_{\mathcal{M}_2}(R)_{(cs,es)} = 0 & \text{for } s \in \mathbb{Z} \end{cases}$$

In the cases where ψ is an isomorphism, some of the results of the chapter can be extended. For instance, if R is Cohen-Macaulay and G is quasi-Gorenstein, for any c, e such that ψ is an isomorphism and $\frac{n}{c} = \frac{a-1}{e} \in \mathbb{Z}$ the ring $k[(I^e)_c]$ is quasi-Gorenstein.

Chapter 5

The a-invariants of the powers of an ideal

Our aim in this chapter is to study in more detail the bigraded a-invariant and the bigraded regularity of any finitely generated bigraded S-module L, for $S = k[X_1, \dots, X_n, Y_1, \dots, Y_r]$ the polynomial ring with $\deg(X_i) = (1, 0)$, $\deg(Y_j) = (0, 1)$.

In Section 5.1 we will give a new description of the a_*-invariant $\mathbf{a}_*(L)$ of L and the regularity $\mathbf{reg}(L)$ of L by means of the a_*-invariants and the regularities of the graded S_1-modules L^e and the graded S_2-modules L_e.

This result is used in Section 5.2 to study the behaviour of the a_*-invariant of the powers of a homogeneous ideal in the polynomial ring. In particular, we will bound it for several families of ideals such as equimultiple ideals and strongly Cohen-Macaulay ideals. Those results will be then applied to determine Cohen-Macaulay diagonals of their Rees algebras.

The last section is devoted to study the regularity of homogeneous ideals I in the polynomial ring S. First, we will provide a bigraded version of the well-known Bayer-Stillman's Theorem characterizing the regularity of I in terms of generic forms. After that, similarly to the graded case, we define the bigraded generic initial ideal $\mathbf{gin}\, I$ of I and we establish its basic properties. In the graded case, a classical result due to D. Bayer and M. Stillman states the existence of an order such that for any homogeneous ideal I it holds $\operatorname{reg} I = \operatorname{reg}(\operatorname{gin} I)$. We will show that the analogous bigraded statement is not true.

We finish the chapter by explaining how these results can be used to study the Koszulness of the diagonals $k[(I^e)_c]$.

5.1 The a-invariant of a standard bigraded algebra

Let $S = k[X_1, \dots, X_n, Y_1, \dots, Y_r]$ be the polynomial ring over a field k in $n + r$ variables with $\deg(X_i) = (1, 0)$, $\deg(Y_j) = (0, 1)$, and let us distinguish two bigraded subalgebras: $S_1 = k[X_1, \dots, X_n]$, $S_2 = k[Y_1, \dots, Y_r]$, with homogeneous maximal ideals $\mathfrak{m}_1 = (X_1, ..., X_n)$, $\mathfrak{m}_2 = (Y_1, \dots, Y_r)$ respectively. Given $e \in \mathbb{Z}$ and a bigraded S-module L, recall that we may define the graded S_1-module $L^e = \bigoplus_{i\in\mathbb{Z}} L_{(i,e)}$ and the graded S_2-module $L_e = \bigoplus_{j\in\mathbb{Z}} L_{(e,j)}$.

The first result shows how to compute the bigraded a_*-invariant of any finitely generated bigraded S-module L by means of the a_*-invariants of the graded S_1-modules L^e and the graded S_2-modules L_e. Namely,

Theorem 5.1.1 *Let L be a finitely generated bigraded S-module. Then :*

(i) $a^1_*(L) = \max_e\{a_*(L^e)\} = \max_e\{a_*(L^e) \mid e \le a^2_*(L) + r\}$.

(ii) $a^2_*(L) = \max_e\{a_*(L_e)\} = \max_e\{a_*(L_e) \mid e \le a^1_*(L) + n\}$.

Proof. Let us consider

$$0 \to D_t \to \dots \to D_1 \to D_0 \to L \to 0$$

the minimal bigraded free resolution of L over S, where $D_p = \bigoplus_{(a,b)\in\Omega_p} S(a, b)$. We have $a^1_*(L) = \max\{-a \mid (a, b) \in \Omega_L\} - n$ by Theorem 1.3.4.

Let us denote by $\underline{\beta} = (\beta_1, \dots, \beta_r) \in \mathbb{N}^r$ and $\mid \underline{\beta} \mid= \beta_1 + \cdots + \beta_r$. By applying the functor $(\ )^e$ to the resolution note that

$$\begin{aligned} S(a,b)^e &= \textstyle\bigoplus_{i\in\mathbb{Z}} S(a,b)_{(i,e)} = \bigoplus_{i\in\mathbb{Z}} S_{(a+i,b+e)} \\ &= \textstyle\bigoplus_{i\in\mathbb{Z}} \bigoplus_{|\underline{\beta}|=b+e} [S_1]_{a+i} Y_1^{\beta_1} \dots Y_r^{\beta_r} \\ &= S_1(a)^{\rho^e_{ab}} \end{aligned}$$

for certain $\rho^e_{ab} \in \mathbb{Z}$ ($\rho^e_{ab} = 0$ if $b + e < 0$). In this way, we have obtained a graded free resolution of L^e over S_1

$$0 \to D^e_t \to \dots \to D^e_1 \to D^e_0 \to L^e \to 0\,,$$

with $D_p^e = \bigoplus_{(a,b)\in\Omega_p} S_1(a)^{\rho_{ab}^e}$. The minimal graded free resolution of L^e may be obtained by picking out some terms [Eis, Exercise 20.1]. Therefore,

$$a_*(L^e) \le \max\{-a \mid (a,b)\in\Omega_L\} - n = a_*^1(L).$$

Now let $\alpha = \max\{-a \mid (a,b)\in\Omega_L\}$. Let p be the first place in the resolution of L with a shift of the form $(-\alpha, b)$, and let β be one of these $-b$'s. We are done if we prove that $-\alpha$ is a shift which appears in the place p of the minimal graded free resolution of L^β. Note that it is enough to show that

$$\operatorname{Tor}_p^S(S/\mathfrak{m}_1 S, L)_{(\alpha,\beta)} = \operatorname{Tor}_p^{S_1}(k, L^\beta)_\alpha \ne 0.$$

Let us consider

$$D_{p+1} \xrightarrow{\psi_{p+1}} D_p \xrightarrow{\psi_p} D_{p-1}$$

the differential maps appearing in the resolution of L. Tensorazing by $S/\mathfrak{m}_1S$, we have the sequence

$$D_{p+1}/\mathfrak{m}_1 D_{p+1} \xrightarrow{\overline{\psi}_{p+1}} D_p/\mathfrak{m}_1 D_p \xrightarrow{\overline{\psi}_p} D_{p-1}/\mathfrak{m}_1 D_{p-1}\ .$$

Now let us take $v\in D_p$ one of the elements of the homogeneous basis of D_p as free S-module with $\deg(v) = (\alpha,\beta)$. If $w_1,\dots,w_s$ is the homogeneous basis of D_{p-1}, we can write

$$\psi_p(v) = \sum_{j=1}^s \lambda_j w_j,$$

with $\lambda_j \in \mathcal{M}$ homogeneous. Set $\deg(w_j) = (\alpha_j, \beta_j)$. By taking into account the way we have choosen α and p, we have $\alpha > \alpha_j$ for any j. Therefore the first component of the degree of λ_j is positive, so $\lambda_j \in \mathfrak{m}_1 S$. We conclude $\overline{\psi}_p(v) = 0$, that is, $v \in \operatorname{Ker} \overline{\psi}_p$. Furthermore, notice that $v \notin \operatorname{Im} \overline{\psi}_{p+1}$ because $\operatorname{Im} \overline{\psi}_{p+1} \subset \mathcal{M}(D_p/\mathfrak{m}_1 D_p)$. So $v \in \operatorname{Tor}_p^S(S/\mathfrak{m}_1S, L)_{(\alpha,\beta)}$, $v \ne 0$. By symmetry, we get (ii). $\square$

Next we are going to consider the bigraded regularity of a finitely generated bigraded S-module L. Assume that

$$0 \to D_t \to \dots \to D_1 \to D_0 \to L \to 0,$$

with $D_p = \bigoplus_{(a,b)\in\Omega_p} S(a,b)$, is the minimal bigraded free resolution of L over S. The bigraded regularity of L is defined by $\mathbf{reg}(L) = (\operatorname{reg}_1 L, \operatorname{reg}_2 L)$, where

$$\operatorname{reg}_1 L = \max_p \{-a-p : (a,b)\in\Omega_p\}$$

$$\operatorname{reg}_2 L = \max_p \{-b-p : (a,b) \in \Omega_p\}.$$

Let $A = k[X_1, \ldots, X_n]$ be the polynomial ring with the usual grading. For any finitely generated graded A-module L, it is well known that

$$\operatorname{reg}(L) = \max_{p \geq 0} \{\, t_p(L) - p\} = \max_{p \geq 0} \{\, a_p(L) + p\}.$$

This equality does not hold in the bigraded case. For instance, let us consider $f_1, \ldots, f_r \in A$ a regular sequence of forms in degree d, and $I = (f_1, \ldots, f_r)$. Let $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ be the polynomial bigraded by setting $\deg(X_i) = (1,0)$, $\deg(Y_j) = (d,1)$, and let R be the Rees algebra of I. Since R is Cohen-Macaulay, we immediately get $a^2_{n+1}(R) = -1$, $a^2_i(R) = 0$ for $i \neq n+1$. Furthermore, the Eagon-Northcott complex gives the bigraded minimal free resolution of R over S:

$$0 \to D_{r-1} \to \ldots \to D_0 = S \to R \to 0,$$

with $D_p = \bigoplus_{m=1}^{p} S(-(p+1)d, -m)^{\binom{r}{p+1}}$ for $p \geq 1$. Therefore,

$$\max_{p \geq 0} \{\, t^2_p(R) - p\} = 0$$

$$\max_{p \geq 0} \{\, a^2_p(L) + p\} = n,$$

which are different.

The following result shows that the regularity of L can also be described by means of the regularity of the graded S_1-modules L^e and the graded S_2-modules L_e. Namely,

Theorem 5.1.2 *Let L be a finitely generated bigraded S-module. Then :*

(i) $\operatorname{reg}_1(L) = \max_e\{\operatorname{reg}(L^e)\} = \max_e\{\operatorname{reg}(L^e) \mid e \leq a^2_*(L) + r\}$.

(ii) $\operatorname{reg}_2(L) = \max_e\{\operatorname{reg}(L_e)\} = \max_e\{\operatorname{reg}(L_e) \mid e \leq a^1_*(L) + n\}$.

Proof. The proof follows the same lines as Theorem 5.1.1. By applying the functor $(\ )^e$ to the minimal bigraded free resolution of L over S, we obtain a graded free resolution of L^e over S_1

$$0 \to D^e_t \to \ldots \to D^e_1 \to D^e_0 \to L^e \to 0\,,$$

with $D_p^e = \bigoplus_{(a,b)\in\Omega_p} S_1(a)^{\rho^e_{ab}}$. Since the minimal graded free resolution of L^e is then obtained by picking out some terms, we have

$$\operatorname{reg}(L^e) \le \max_p\{-a-p \mid (a,b)\in\Omega_p\} = \operatorname{reg}_1 L.$$

Hence $\max_e\{\operatorname{reg}(L^e)\} \le \operatorname{reg}_1 L$. To prove the equality, let us take $(a,b)\in\Omega_p$ such that $\operatorname{reg}_1 L = -a-p$, and set $\alpha=-a$, $\beta=-b$. We are done if we prove $a\in\Omega_{p,L^\beta}$, that is, a is a shift which appears in the place p of the minimal graded free resolution of L^β. So we want to show that

$$\operatorname{Tor}_p^S(S/\mathfrak{m}_1 S, L)_{(\alpha,\beta)} = \operatorname{Tor}_p^{S_1}(k, L^\beta)_\alpha \neq 0.$$

Let us consider

$$D_{p+1} \xrightarrow{\psi_{p+1}} D_p \xrightarrow{\psi_p} D_{p-1}$$

the differential maps appearing in the resolution of L. Tensorazing by $S/\mathfrak{m}_1 S$, we have the sequence

$$D_{p+1}/\mathfrak{m}_1 D_{p+1} \xrightarrow{\overline{\psi}_{p+1}} D_p/\mathfrak{m}_1 D_p \xrightarrow{\overline{\psi}_p} D_{p-1}/\mathfrak{m}_1 D_{p-1}.$$

Now let $v\in D_p$ be an element of the homogeneous basis of D_p as free S-module with $\deg(v)=(\alpha,\beta)$. If $w_1,\ldots,w_s$ is the homogeneous basis of D_{p-1}, we can write

$$\psi_p(v) = \sum_{j=1}^s \lambda_j w_j,$$

with $\lambda_j\in\mathcal{M}$ homogeneous. Set $\deg(w_j)=(\alpha_j,\beta_j)$. Since $\alpha-p\ge\alpha_j-(p-1)$ for any j, we have that $\alpha>\alpha_j$, and so the first component of the degree of λ_j is positive. Therefore $\lambda_j\in\mathfrak{m}_1 S$, and we can conclude $\overline{\psi}_p(v)=0$, that is, $v\in\operatorname{Ker}\overline{\psi}_p$. It is clear that $v\notin\operatorname{Im}\overline{\psi}_{p+1}$ because $\operatorname{Im}\overline{\psi}_{p+1}\subset\mathcal{M}(D_p/\mathfrak{m}_1 D_p)$. So $v\in\operatorname{Tor}_p^S(S/\mathfrak{m}_1 S, L)_{(\alpha,\beta)}$, $v\neq 0$. We get (ii) by symmetry. $\square$

5.2 The a-invariants of the powers of an ideal

Let $A=k[X_1,\ldots,X_n]$ be the usual polynomial ring over a field k, and let I be a homogeneous ideal in A. Recently, the question of how the regularity changes with the powers of I has been studied by many authors. I. Swanson in [Swa] proved that there exists an integer B such that $\operatorname{reg}(I^e)\le Be$ for all e. The problem is then to make B explicit.

For ideals such that $\dim(A/I) = 1$, A. Geramita, A. Gimigliano and Y. Pitteloud [GGP] and K. Chandler [Cha] had shown that $\operatorname{reg}(I^e) \leq \operatorname{reg}(I)\, e$; and this bound also holds for Borel-fixed monomial ideals by using the Eliahou-Kervaire resolution [EK].

Another kind of bound is given by R. Sjögren [Sjo]: If I is an ideal generated by forms in degree $\leq d$ with $\dim(A/I) \leq 1$, then $\operatorname{reg}(I^e) < (n-1)de$. Also A. Bertram, L. Ein and R. Lazarsfeld [BEL] gave a bound for the regularity of the powers of an ideal in terms of the degrees of its generators. More explicitly, if I is the ideal of a smooth complex subvariety X in $\mathbb{P}^{n-1}_{\mathbb{C}}$ of codimension c and I is generated by forms in degrees $d_1 \geq d_2 \geq \ldots \geq d_r$, then

$$H^i(\mathbb{P}^{n-1}_{\mathbb{C}}, \mathcal{I}^e(k)) = 0, \ \ \forall i \geq 1, \forall k \geq ed_1 + d_2 + \ldots + d_c - (n-1).$$

This result has been improved by A. Bertram [Ber] for some determinantal varieties.

Recently, work by S.D. Cutkosky, J. Herzog and N.V. Trung [CHT], V. Kodiyalam [Ko2] and O. Lavila–Vidal (see Theorem 3.4.6) provides by different methods bounds for arbitrary graded ideals by means of the degrees of the generators similar to the ones given in [Sjo] and [BEL]. Namely, if I is a graded ideal generated by forms in degree $\leq d$, then there exists β such that

$$\operatorname{reg}(I^e) \leq de + \beta, \ \forall e.$$

We are also interested in the behaviour of the a_*-invariant of the powers of I, which can be used to apply the criteria seen in Chapter 3 for the Cohen-Macaulayness of the diagonals. We have already proved in Theorem 3.4.6 the existence of an integer α such that $a_*(I^e) \leq de + \alpha$ for all e. Our first purpose will be to find for any graded ideal an explicit α. Furthermore, for equigenerated ideals we will compute the best α we can take in terms of an appropiate a-invariant of the Rees algebra. After that, these results will be applied to give bounds for the a_*-invariant of the powers of several families of ideals such as equimultiple ideals and strongly Cohen-Macaulay ideals. Finally, we will use those bounds to study the Cohen-Macaulay property of the diagonals of the Rees algebra.

Let k be a field, A a standard noetherian graded k-algebra, $\overline{n} = \dim A$. Then A has a presentation $A = k[X_1, \ldots, X_n]/K = k[x_1, \ldots, x_n]$, where K is a homogeneous ideal and each X_i has degree 1. Let I be a homogeneous ideal

in A generated by forms of degree $\leq d$. From Theorem 3.4.6, there exists α such that

$$a_*(I^e) \leq de + \alpha,\ \forall e.$$

Now let us assume that I is generated by forms in degree d. By defining $\varphi(p,q) = (p - dq, q)$, we have that R^φ is a standard bigraded k-algebra with $[R^\varphi]_{(p,q)} = R_{(p+dq,q)}$. The next result precises the best α we can take.

Theorem 5.2.1 *Let I be a homogeneous ideal of A generated by forms in degree d. Set $l = l(I)$. Then*

(i) $a^1_*(R^\varphi) = \max_e\{\, a_*(I^e) - de\} = \max\{\, a_*(I^e) - de \mid e \leq a^2_*(R) + l\}$.

(ii) $\operatorname{reg}_1(R^\varphi) = \max_e\{\operatorname{reg}(I^e) - de\} = \max\{\operatorname{reg}(I^e) - de \mid e \leq a^2_*(R) + l\}$.

Proof. We may assume that k is infinite (tensorazing by $k(T)$). Then there exists a minimal reduction J of I generated by l forms in degree d. By considering the polynomial ring $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_l]$, we have a natural epimorhism $S \to R_A(J)$. Then $R_A(J)$ is a finitely generated bigraded S-module, and so $R = R_A(I)$ because it is a finitely generated $R_A(J)$-module. Note that S^φ is standard and R^φ is a finitely generated bigraded S^φ-module, so according to Theorem 5.1.1

$$a^1_*(R^\varphi) = \max_e\{a_*([R^\varphi]^e)\} = \max_e\{a_*([R^\varphi]^e) \mid e \leq a^2_*(R^\varphi) + l\}.$$

First, observe that $a^2_*(R^\varphi) = a^2_*(R)$ by Lemma 1.2.3. Moreover, for each $e \geq 0$ we have $[R^\varphi]^e = \bigoplus_i (I^e)_{i+de} = I^e(de)$, so $a_*((I^e)^\psi) = a_*(I^e) - de$. The proof of (ii) follows the same lines. $\square$

Remark 5.2.2 Let I be a homogeneous ideal in A generated by forms in degree d. By repeating the previous arguments for the form ring, we also get

(i) $a^1_*(G^\varphi) = \max_e\{\, a_*(I^e/I^{e+1}) - de\}$
$= \max_e\{\, a_*(I^e/I^{e+1}) - de \mid e \leq a^2_*(G) + l\}$.

(ii) $\operatorname{reg}_1(G^\varphi) = \max_e\{\operatorname{reg}(I^e/I^{e+1}) - de\}$
$= \max_e\{\operatorname{reg}(I^e/I^{e+1}) - de \mid e \leq a^2_*(G) + l\}$.

Example 5.2.3 Let $I \subset A = k[X_1, X_2, X_3, X_4]$ be the defining ideal of the twisted cubic in $\mathbb{P}^3_k$, that is,

$$I = (X_1X_4 - X_2X_3, X_2^2 - X_1X_3, X_3^2 - X_2X_4).$$

It is well known that I is the ideal of the Veronese embedding of $\mathbb{P}^1_k$ in $\mathbb{P}^3_k$:

$$\begin{array}{ccc} \mathbb{P}^1_k & \xrightarrow{\mu} & \mathbb{P}^3_k \\ (u:v) & \longmapsto & (u^3 : u^2v : uv^2 : v^3) \,. \end{array}$$

I is licci because it is linked to $J = (X_1, X_2)$ by the regular sequence $\underline{\alpha} = X_2^2 - X_1X_3, X_3^2 - X_2X_4$ [Ul, Example 2.3], so I is a strongly Cohen-Macaulay ideal [Hu1, Theorem 1.14]. Since I is a prime ideal, we easily get $\mu(I_\mathfrak{p}) \leq \mathrm{ht}\,(\mathfrak{p})$ for any prime ideal $\mathfrak{p} \supseteq I$. Therefore $R_A(I)$ is Cohen-Macaulay by [HSV1, Theorem 2.6], so in particular $a^2_*(R_A(I)) = -1$. On the other hand, I is an ideal generated by forms of degree 2 with $l(I) = \mu(I) = 3$. By using CoCoa [CNR], we have that the minimal graded free resolutions of I and I^2 are:

$$0 \to A(-3)^2 \to A(-2)^3 \to I \to 0 \ ,$$

$$0 \to A(-6) \to A(-5)^6 \to A(-4)^6 \to I^2 \to 0 \ ,$$

so according to Theorem 1.3.4 we have $a_*(I) = -1$, $a_*(I^2) = 2$. By Theorem 5.2.1 we get

$$a_*(I^e) \leq 2(e-1), \ \forall e.$$

Furthermore, notice that since $\mathrm{reg}\,(I) = 2$, $\mathrm{reg}\,(I^2) = 4$ we also get

$$\mathrm{reg}\,(I^e) \leq 2e, \ \forall e.$$

Therefore, we have that I^e has a linear resolution for any $e \geq 1$. This has already been proved by A. Conca [Con] by different methods.

Remark 5.2.4 Let $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ be the polynomial ring bigraded by setting $\deg(X_i) = (1,0)$, $\deg(Y_j) = (d_j, 1)$, with $d_1, \ldots, d_r \in \mathbb{Z}_{\geq 0}$, and $u = \sum_{j=1}^r d_j$. For a finitely generated bigraded S-module L, let us consider the minimal bigraded free resolution of L over S

$$0 \to D_t \to \ldots \to D_1 \to D_0 \to L \to 0,$$

where $D_p = \bigoplus_{(a,b)\in\Omega_p} S(a,b)$. By applying the functor $(\ )^e$, note that

$$S(a,b)^e = \bigoplus_{|\underline{\beta}|=b+e} S_1(a - d_1\beta_1 - \ldots - d_r\beta_r),$$

where $\underline{\beta} = (\beta_1, \ldots, \beta_r) \in \mathbb{N}^r$ and $|\ \underline{\beta}\ | = \beta_1 + \cdots + \beta_r$. So we get a graded free resolution of L^e over S_1

$$0 \to D^e_t \to \ldots \to D^e_1 \to D^e_0 \to L^e \to 0\,,$$

with $D_p^e = \bigoplus_{(a,b)\in\Omega_p} \bigoplus_{|\underline{\beta}|=b+e} S_1(a - d_1\beta_1 - \ldots - d_r\beta_r)$. The minimal graded free resolution is then obtained by picking out some terms. Therefore, for any $i \leq n$ we have that

$$a_i(L^e) \leq \max\{d_1\beta_1 + \ldots + d_r\beta_r - a \mid (a,b) \in \Omega_{n-i}, \mid \underline{\beta} \mid = b+e\} - n$$

$$\leq de - n + \max\{db - a \mid (a,b) \in \Omega_{n-i}\}.$$

Therefore, $a_*(L^e) \leq de - n + \max\{db - a \mid (a,b) \in \Omega_L\} \leq d(e - \mathrm{indeg}_2 L) + a_*^1(L) + u$. In particular, for any homogeneous ideal I of A we have

$$a_*(I^e) \leq de + a_*^1(R) + u.$$

5.2.1 Explicit bounds for some families of ideals

The next purpose is to get explicit bounds for the a_*-invariant of the powers of an ideal, and we will focus our attention to the case of ideals in the polynomial ring. Throughout the rest of this section, $A = k[X_1, \ldots, X_n]$ will denote the usual polynomial ring in n variables over a field k and I will be a homogeneous ideal in A. First of all, for equigenerated ideals whose Rees algebra is Cohen-Macaulay we have

Proposition 5.2.5 *Let I be a homogeneous ideal generated by forms in degree d whose Rees algebra is Cohen-Macaulay. Set $l = l(I)$. Then*

$$-n + d(-a^2(G) - 1) \leq \max_{e\geq 0}\{a_*(I^e) - de\} \leq -n + d(l-1).$$

Proof. As in the proof of Theorem 5.2.1, we may assume that k is infinite. By considering then the polynomial ring $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_l]$, we have that $R_A(I)$ is a finitely generated bigraded S-module in a natural way. Then let

$$0 \to D_t \to \ldots \to D_1 \to D_0 \to R \to 0$$

be the minimal bigraded free resolution of the Rees algebra R over S, with $D_p = \bigoplus_{(a,b)\in\Omega_p} S(a,b)$. The shifts $(a,b) \in \Omega_R$, $(a,b) \neq (0,0)$, satisfy $b \leq -1$ and $-a \leq dl + n + a^1(R) \leq dl$ by Lemma 3.4.7. Therefore, we have $a_*(I^e) \leq de + d(l-1) - n$ by Remark 5.2.4.

Let $R_{++} = \bigoplus_{(i,j),j>0} R_{(i,j)}$. From the bigraded exact sequences

$$0 \to R_{++} \to R \to A \to 0\,,$$

$$0 \to R_{++}(0,1) \to R \to G \to 0\,,$$

we get the following exact sequences of local cohomology

$$0 \to H^n_{\mathfrak{m}}(A)_{(i,j)} \to H^{n+1}_{\mathcal{M}}(R_{++})_{(i,j)} \to H^{n+1}_{\mathcal{M}}(R)_{(i,j)} \to 0 \quad (\star),$$

$$0 \to H^n_{\mathcal{M}}(G)_{(i,j)} \to H^{n+1}_{\mathcal{M}}(R_{++})_{(i,j+1)} \to H^{n+1}_{\mathcal{M}}(R)_{(i,j)} \to 0 \quad (\star\star).$$

Since $a^2(R) = -1$, from the above exact sequences we have $a^2(G) \leq -1$. If $a^2(G) = -1$, the lower bound is obvious by considering $e = 0$. So we may assume $a^2(G) < -1$, and by Theorem 5.2.1 we must prove $a^1(R^\varphi) \geq -n-d(a^2(G)+1)$. The local cohomology modules behave well under a change of grading by Lemma 1.2.3, hence we have

$$H^{n+1}_{\mathcal{M}}(R^\varphi)_{(p,q)} = H^{n+1}_{\mathcal{M}}(R)^\varphi_{(p,q)} = H^{n+1}_{\mathcal{M}}(R)_{(p+dq,q)},$$

so $a^1(R^\varphi) = \max\{\, p \mid \exists\ q\ s.t.\, H^{n+1}_{\mathcal{M}}(R)_{(p+dq,q)} \neq 0\}$. Since $H^n_{\mathfrak{m}}(A)_{(-n,0)} \neq 0$ we have $H^{n+1}_{\mathcal{M}}(R_{++})_{(-n,0)} \neq 0$ from the exact sequence $(\star)$. As $a^2(G) < -1$, from the second exact sequence $(\star\star)$ we get $H^{n+1}_{\mathcal{M}}(R)_{(-n,-1)} \neq 0$, and by using once more $(\star)$ we have $H^{n+1}_{\mathcal{M}}(R_{++})_{(-n,-1)} \neq 0$. Note that we can repeat this procedure while the second component of the degree be greater than $a^2(G)$, and finally we get $H^{n+1}_{\mathcal{M}}(R)_{(-n,a^2(G)+1)} \neq 0$. In particular, $a^1(R^\varphi) \geq -n-d(a^2(G)+1)$. $\square$

Remark 5.2.6 Let I be an ideal generated by forms of degree d in a general standard graded noetherian k-algebra A. By setting $l = l(I)$, one can similarly prove that if the Rees algebra is Cohen-Macaulay then

$$a(A) + d(-a^2(G) - 1) \leq \max\{a_*(I^e) - de\} \leq a(A) + dl.$$

For non-equigenerated ideals, we can also give an upper bound. A similar result for the regularity was already proved in [CHT, Corollary 2.6].

Remark 5.2.7 Let I be a homogeneous ideal generated by forms $f_1, \ldots, f_r$ in degrees $d_1 \leq \ldots \leq d_r = d$ whose Rees algebra is Cohen-Macaulay. Set $u = \sum_{i=1}^r d_i$. Then $a_*(I^e) \leq d(e-1) + u - n$.

Proof. By considering the bigraded minimal free resolution of R over $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$, we have that any shift $(a,b) \in \Omega_p$ with $p \geq 1$ satisfies $b \leq -1$ and $-a \leq u$ by Lemma 3.4.10 and Ω_0 only contains the shift $(0,0)$. Therefore, $a_*(I^e) \leq d(e-1) + u - n$ by Remark 5.2.4. $\square$

The a_*-invariant of the powers of an ideal can be computed for complete intersection ideals (see the proof of Proposition 3.4.5), and then we have that the inequalities in Proposition 5.2.5 and Remark 5.2.7 are sharp. Next we are going to compute explicitly $\max_{e\geq 0}\{a_*(I^e)-de\} = a^1(R^\varphi)$ for several families of ideals. First we consider the case of equimultiple ideals.

Proposition 5.2.8 *Let I be an equimultiple ideal generated in degree d. Set $h = \mathrm{ht}\,(I) \geq 1$. If the Rees algebra is Cohen-Macaulay,*

(i) $a(I^e/I^{e+1}) = de + a(A/I)$. In particular, $a^1(G^\varphi) = a(A/I)$.

(ii) $a_{n-h+1}(I^e) = d(e-1) + a(A/I)$. In particular, $a^1(R^\varphi) = a(A/I) - d$.

Proof. We may assume that k is infinite. Then there exist $f_1, \dots, f_{n-h} \in A$ of degree 1 such that $\overline{f}_1, \dots, \overline{f}_{n-h} \in A/I$ is a homogeneous system of parameters. Denoting by f^* the initial form of $f \in A$ in G, let us consider $\overline{G} = G/(f_1^*, \dots, f_{n-h}^*)$. Since $\mathrm{rad}\,((f_1^*, \dots, f_{n-h}^*)G) = \mathrm{rad}\,(\mathfrak{m}G)$, we have that a system of parameters of $F_{\mathfrak{m}}(I)$ is also a system of parameters of $\overline{G}$. As $F_{\mathfrak{m}}(I)$ is a bigraded k-algebra generated by forms in degree $(d,1)$, there exist $F_1, \dots, F_h \in I$ of degree d such that $\overline{F}_1, \dots, \overline{F}_h$ is a system of parameters of $F_{\mathfrak{m}}(I)$. Then $f_1^*, \dots, f_{n-h}^*, F_1^*, \dots, F_h^*$ is a homogeneous system of parameters of G, and so algebraically independent over k. Therefore, there is a finite extension

$$T = k[U_1, \dots, U_h, V_1, \dots, V_{n-h}] \to G,$$

where T is a polynomial ring with $\deg(U_i) = (d,1)$, $\deg(V_j) = (1,0)$ for $i = 1, \dots, h$, $j = 1, \dots, n-h$. Since G is Cohen-Macaulay and T is regular, we have that G is a free T-module, that is, $G = \bigoplus_{(a,b)\in\Lambda} T(a,b)$, where $\Lambda \subset \mathbb{Z}^2$ is a finite set. Let us denote by $T_1 = k[V_1, \dots, V_{n-h}]$, $m = (V_1, \dots, V_{n-h})$, and for a given $\underline{\beta} = (\beta_1, \dots, \beta_h) \in \mathbb{N}^h$, let $\mid \underline{\beta} \mid = \beta_1 + \cdots + \beta_h$. Note that

$$\begin{aligned} T(a,b)^e &= \textstyle\bigoplus_i T(a,b)_{(i,e)} = \bigoplus_i T_{(a+i,b+e)} \\ &= \textstyle\bigoplus_i \bigoplus_{|\underline{\beta}|=b+e} [T_1]_{a+i-d(b+e)} U_1^{\beta_1} \dots U_h^{\beta_h} \\ &= T_1(a-db-de)^{\rho_b^e}, \end{aligned}$$

with $\rho_b^e \in \mathbb{N}$ and $\rho_b^e = 0$ if $b+e<0$. Therefore,

$$I^e/I^{e+1} = G^e = \bigoplus_{(a,b)\in\Lambda} T_1(a-db-de)^{\rho_b^e},$$

and by taking local cohomology

$$H_{\mathfrak{m}}^{n-h}(I^e/I^{e+1}) = \bigoplus_{(a,b)\in\Lambda} H_m^{n-h}(T_1(a-db-de))^{\rho_b^e}.$$

Hence $a_*(I^e/I^{e+1}) = a(I^e/I^{e+1}) = \max\{-(n-h)-a+db+de : -b \leq e\}$. In particular, $a(A/I) = \max\{-(n-h)-a+db : b = 0\}$, and so we get $a(I^e/I^{e+1}) \geq de + a(A/I)$ for all e. On the other hand, since the modules I^e/I^{e+1} are A/I-modules of maximal dimension, we have an epimorhism $\bigoplus A/I(-de) \to I^e/I^{e+1}$ and we may deduce that $a(I^e/I^{e+1}) \leq de + a(A/I)$ for all e. To get (ii), it is just enough to consider the short exact sequences

$$0 \to I^{e+1} \to I^e \to I^e/I^{e+1} \to 0\ ,$$

and then the result follows from (i) by induction on e. $\square$

Next we study equigenerated ideals whose form ring is Gorenstein. In this case, we prove that the lower bound given in Proposition 5.2.5 is sharp.

Proposition 5.2.9 *Let I be a homogeneous ideal equigenerated in degree d whose form ring is Gorenstein. Set $l = l(I)$. Then*

(i) $\max_{e\geq 0}\{a_*(I^e) - de\} = d(-a^2(G)-1) - n$.

(ii) For $e > a^2(G) - a(F)$, $\operatorname{depth}(A/I^e) = n - l$ *and* $a_*(I^e) = a_{n-l}(A/I^e) = d(e - a^2(G) - 1) - n$.

Proof. We may assume that the field k is infinite. Since I is generated by forms in degree d, there exists a minimal reduction J of I generated by forms $g_1, \dots, g_l$ of degree d. By considering $S = k[X_1, \dots, X_n, Y_1, \dots, Y_l]$ bigraded by setting $\deg(X_i) = (1, 0)$, $\deg(Y_j) = (d, 1)$, we have a bigraded epimorphism $S \to R_A(J)$. Suppose that $I^{m+1} = JI^m$. Then $R_A(I)$ is finitely generated over $R_A(J)$ by the generators of A, I, $\dots$, I^m; so in particular by homogeneous elements in degree (di, i) for $i = 0, \dots, m$. Then we have an epimorphism $F \to R_A(I)$, where F is a finite free S-module with a basis of elements in degrees (di, i) for $i = 0, \dots, m$.

Let us consider the minimal bigraded free resolution of G over S

$$0 \to D_l \to \dots \to D_1 \to D_0 \to G \to 0,$$

where $D_p = \bigoplus_{(a,b)\in\Omega_p} S(a, b)$. From Remark 5.2.4, for all $e \geq 0$ we have

$$a_*(I^e/I^{e+1}) \leq de - n + \max\{db - a \mid (a, b) \in \Omega_G\}.$$

Assume we prove that the maximum is accomplished for a shift $(a,b) \in \Omega_l$. Denoting by $(\ )^* = \underline{\mathrm{Hom}}_S(\ ,K_S)$, then

$$0 \to D_0^* \to \ldots \to D_l^* \to \underline{\mathrm{Ext}}_S^l(G,K_S) = K_G \to 0$$

is the minimal bigraded free resolution of the canonical module K_G of G over S. Since G is Gorenstein, according to Corollary 4.1.7 there is a bigraded isomorphism

$$K_G \cong G(-n, a^2(G)).$$

Now the shifts $(a,b) \in \Omega_l$ are of the type $(di, i - a^2(G))$ for certain integers i, so we get $a_*(I^e/I^{e+1}) \le de - n + \max_i\{d(i - a^2(G)) - di\} = d(e - a^2(G)) - n$. From Remark 5.2.5 we have $a^1(G^\varphi) = \max\{a_*(I^e/I^{e+1}) - de\} \le -n - da^2(G)$, and $a^1(R^\varphi) \le -n - da^2(G) - d$. Observe that Proposition 5.2.5 gives the other inequality, so we obtain $a^1(R^\varphi) = d(-a^2(G) - 1) - n$.

Now let us prove that the maximum for the differences $db - a$ is taken for $(a,b) \in \Omega_l$. Let us consider $\phi : \mathbb{Z}^2 \to \mathbb{Z}$ defined by $\phi(i,j) = i - dj$. Note that $X_i \in S^\phi$ has degree 1 and $Y_j \in S^\phi$ has degree 0, and

$$0 \to D_l^\phi \to \ldots \to D_0^\phi \to G^\phi \to 0$$

is a graded free resolution of G^ϕ over S^ϕ. Since $S(a,b)^\phi = S^\phi(a - db)$, it is clear that

$$\min\{db - a \mid (a,b) \in \Omega_{p+1}\} \ge \min\{db - a \mid (a,b) \in \Omega_p\}.$$

Applying the same argument for the resolution of K_G, one gets that

$$\max\{db - a \mid (a,b) \in \Omega_{p+1}\} \ge \max\{db - a \mid (a,b) \in \Omega_p\},$$

so (i) is already proved.

To prove the rest of the statement, we will use that $\mathrm{proj.dim}_A(I^e/I^{e+1}) = l$ if and only if $e \ge a^2(G) - a(F)$ (see Proposition 6.3.2). By applying the functor $(\ )^e$ to the resolution of G over S we obtain a free resolution of I^e/I^{e+1} as A-module, and the shifts appearing in the place p of these resolutions of the type $a - db - de$, with $(a,b) \in \Omega_p$. Therefore, $t_p(I^e/I^{e+1}) \le de - da^2(G)$ for any p, and for $e \ge a^2(G) - a(F)$ we have $t_l(I^e/I^{e+1}) = de - da^2(G)$. Now, since $\mathrm{proj.dim}_A I^e \le l - 1$ by Proposition 6.3.2, from the short exact sequences

$$0 \to I^{e+1} \to I^e \to I^e/I^{e+1} \to 0,$$

we get $t_{l-1}(I^e) \geq d(e - a^2(G) - 1)$ for $e > a^2(G) - a(F)$. On the other hand, we have $t_{l-1}(I^e) \leq t_*(I^e) = a_*(I^e) + n \leq d(e - a^2(G) - 1)$, so we get the equality. We finally obtain $a_{n-l}(A/I^e) = a_{n-l+1}(I^e) = d(e - a^2(G) - 1) - n$ for $e > a^2(G) - a(F)$ by Theorem 1.3.4. $\square$

Example 5.2.10 Let $\mathbf{X} = (X_{ij})$ be a $d \times n$ generic matrix, with $1 \leq i \leq d$, $1 \leq j \leq n$ and $d \leq n$, and let $A = k[\mathbf{X}]$ be the polynomial ring in the entries of $\mathbf{X}$. Let $I = I_d(\mathbf{X})$ be the ideal generated by the maximal minors of $\mathbf{X}$. We are going to apply to this example the different bounds we have found.

- The Rees algebra of I is Cohen-Macaulay, so by applying Remark 5.2.7 we get
$$a_*(I^e) \leq d(e-1) + \binom{n}{d} d - nd.$$
(a similar bound for the regularity of the powers has been given in [CHT, Example 2.7]).

- Note that $F_{\mathfrak{m}}(I) = k[I_d]$ is the coordinate ring of the Grassmannian $G(d, n)$, so we have that the analytic spread of I is $l(I) = d(n - d) + 1$. Therefore by Proposition 5.2.5 we get the bound
$$a_*(I^e) \leq de + d^2(n - d) - nd.$$

- Since $G_A(I)$ is Gorenstein with $a^2(G_A(I)) = -\mathrm{ht}\,(I) = -(n - d + 1)$, by using Proposition 5.2.9 we get the better bound
$$a_*(I^e) \leq d(e + n - d) - nd = de - d^2.$$
Furthermore, the a-invariant of F is $-n$ by [BH2, Corollary 1.4]. So we also obtain $a_{d^2}(I^e) = de - d^2$ for any $e > d - 1$.

K. Akin et al. [ABW] have constructed resolutions for the powers of I, in particular showing that all the powers of I have linear resolutions. Note that this fact also allows to prove the last bound: According to Proposition 6.3.2 and Theorem 1.3.4, for any $e > a^2(G) - a(F)$ we have $a_*(I^e) = a_{nd-l+1}(I^e) = t_{l-1}(I^e) - nd = (de + l - 1) - nd = de - d^2$.

We may also use Proposition 5.2.9 to study the a_*-invariant of the powers of a strongly Cohen-Macaulay ideal. Let I be an ideal of A, and let $\underline{f} = f_1, \ldots, f_r$

be a system of generators of I. Recall that I is a strongly Cohen-Macaulay ideal if for any $p \geq 0$ the Koszul homology $H_p(K(\underline{f}))$ is a Cohen-Macaulay A/I-module.

Corollary 5.2.11 *Let I be a strongly Cohen-Macaulay ideal generated in degree d such that $\mu(I_{\mathfrak{p}}) \leq \mathrm{ht}\,(\mathfrak{p})$ for any prime ideal $\mathfrak{p} \supseteq I$. Let $h = \mathrm{ht}\,(I)$, $l = l(I)$. Then*

(i) $a_*(I^e) \leq d(e+h-1) - n$, $\forall e$.

(ii) For $e > l - h$, $\mathrm{depth}(A/I^e) = n - l$ *and* $a_*(I^e) = a_{n-l}(A/I^e) = d(e+h-1) - n$.

Proof. In this situation, $G_A(I)$ is Gorenstein and I is an ideal of linear type by [HSV1, Theorem 2.6], so $a(F_{\mathfrak{m}}(I)) = -l$. Furthermore, according to [HRZ, Proposition 2.5] we have $a^2(G_A(I)) = -h$. Then the result follows from Proposition 5.2.9.□

Example 5.2.12 Let $I \subset A = k[X_1, X_2, X_3, X_4]$ be the defining ideal of the twisted cubic in $\mathbb{P}^3_k$. From Example 5.2.3, recall that I is a strongly Cohen-Macaulay ideal generated in degree 2 with $\mathrm{ht}(I) = 2$, $l(I) = \mu(I) = 3$. Now, by Corollary 5.2.11, for any $e > 1$ we have that $\mathrm{depth}(A/I^e) = 1$, $a_1(A/I^e) = 2e - 2$ and $a_2(A/I^e) \leq 2e - 2$. In the case $e = 1$, since I is linked to $J = (X_1, X_2)$ by the regular sequence $\underline{\alpha} = X_2^2 - X_1X_3, X_3^2 - X_2X_4$, we have that A/I is Cohen-Macaulay and there is a graded isomorphism $K_{A/I} \cong J/(\underline{\alpha})$. In particular, $a(A/I) = -1$.

In trying to extend the bounds in Proposition 5.2.9 to the non-equigenerated case many difficulties appear. Next we will use approximation complexes to do this for strongly Cohen-Macaulay ideals.

Let I be a homogeneous ideal in the polynomial ring $A = k[X_1, \ldots, X_n]$ and $\underline{f} = f_1, \ldots, f_r$ a homogeneous system of generators of I, with $d_i = \deg(f_i)$, and let us consider the graded Koszul complex $K(\underline{f})$ of A with respect to $\underline{f}$. Denote by $S = A[Y_1, \ldots, Y_r]$ with the bigrading $\deg(X_i) = (1, 0)$, $\deg(Y_j) = (d_j, 1)$. Then the approximation complex of I is

$$\mathcal{M}(\underline{f}) : 0 \to \mathcal{M}_r \to \ldots \to \mathcal{M}_1 \to \mathcal{M}_0 \to 0,$$

with $\mathcal{M}_p = H_p(K(\underline{f})) \otimes_A S(0, -p)$, and the differential maps are homogeneous. Assume that I is a strongly Cohen-Macaulay ideal with $\mathrm{ht}\,(I) \geq 1$ such that

for any prime ideal $\mathfrak{p} \supseteq I$, $\mu(I_{\mathfrak{p}}) \leq \mathrm{ht}\,(\mathfrak{p})$. Then $\mathcal{M}(\underline{f})$ is exact and provides a resolution of $\mathrm{Sym}_A(I/I^2) \cong G_A(I)$ [HSV1, Theorem 2.6]. We will use it to get a bound for the a-invariants of the powers of these ideals.

Proposition 5.2.13 *Let I be a strongly Cohen-Macaulay ideal such that for any prime ideal $\mathfrak{p} \supseteq I$, $\mu(I_{\mathfrak{p}}) \leq \mathrm{ht}\,(\mathfrak{p})$. Assume that I is minimally generated by forms $f_1, \dots, f_r$ of degree $d = d_1 \geq \dots \geq d_r$, and set $h = \mathrm{ht}\,(I)$, $t = r - h$. Then:*

(i) $a(H_m(\underline{f})) \leq -n + d_1 + \dots + d_{h+m}$, *for all* $m \leq t$.

(ii) If $1 \leq e \leq t$, $\mathrm{depth}(A/I^e) \geq n - h - e + 1$ *and for any* $0 \leq m \leq e-1$,

$$a_{n-h+1-m}(I^e) \leq -n + d_1 + \dots + d_{h+m} + d(e-m-1).$$

(iii) If $e > t$, $\mathrm{depth}(A/I^e) = n - r$ *and for any* $0 \leq m \leq t$,

$$a_{n-h+1-m}(I^e) \leq -n + d_1 + \dots + d_{h+m} + d(e-m-1).$$

Proof. Recall that $H_p = H_p(K(\underline{f})) = 0$ for all $p > t$. So the resolution of G given by the approximation complex is

$$0 \to \mathcal{M}_t \to \dots \to \mathcal{M}_0 \to G \to 0,$$

with $\mathcal{M}_p = H_p \otimes_A S(0, -p)$. Let us denote by $\underline{\beta} = (\beta_1, \dots, \beta_r) \in \mathbb{N}^r$, and $\mid \underline{\beta} \mid = \beta_1 + \dots + \beta_r$. Applying the functor $(\ )^e$ to the modules of this resolution:

$$G^e = \bigoplus_i G_{(i,e)} = I^e/I^{e+1},$$

$$\begin{aligned}\mathcal{M}_p^e &= \bigoplus_i H_p[Y_1, \dots, Y_r]_{(i,e-p)} \\ &= \begin{cases} 0 & \text{if } e < p \\ \bigoplus_{|\underline{\beta}|=e-p} H_p(-d_1\beta_1 - \dots - d_r\beta_r) & \text{if } e \geq p. \end{cases}\end{aligned}$$

So we get graded exact sequences

$$0 \to \mathcal{M}_q^e \to \dots \to \mathcal{M}_0^e \to I^e/I^{e+1} \to 0,$$

with $q = \min\{e, t\}$. Since I is a strongly Cohen-Macaulay ideal, we have that $\mathcal{M}_p^e$ is a maximal CM A/I-module for any $p \leq q$. By taking short exact sequences, we obtain that if $e < t$, $\mathrm{depth}(I^e/I^{e+1}) \geq n - h - e$ and if $e \geq t$, $\mathrm{depth}(I^e/I^{e+1}) \geq n - h - t = n - r$. On the other hand, by Proposition 6.3.2 we also have that $\mathrm{depth}(I^e/I^{e+1}) = n - r$ if and only if $e \geq t$. Furthermore, we

get $a_{n-h-m}(I^e/I^{e+1}) \leq a(\mathcal{M}^e_m) = a(H_m)+d(e-m)$ for all $0 \leq m \leq \min\{e,t\}$. From the exact sequences

$$0 \to I^e/I^{e+1} \to A/I^{e+1} \to A/I^e \to 0,$$

we have now

(a) $\operatorname{depth}(A/I^e) \geq n-h-e+1$ if $1 \leq e \leq t$

(b) $\operatorname{depth}(A/I^e) = n-r$ if $e > t$

(c) $a_{n-h-m}(A/I^e) \leq \max_{0\leq j\leq e-1}\{a_{n-h-m}(I^j/I^{j+1})\} \leq a(H_m)+d(e-m-1)$, for $0 \leq m \leq \min\{e-1,t\}$.

So, if we prove the bound for the a-invariant of the Koszul homology we have finished. Let us assume that among the forms $f_1,\dots,f_r$ we can choose a regular sequence of length h. Let $g_1 = f_{j_1},\dots,g_h = f_{j_h}$ be this sequence, and $g_1,\dots g_r$ the minimal system of generators of I.

Let us consider the morphism from A to $A/(g_1)$. By [Hu2, Lemma 1.1], there is a graded exact sequence

$$0 \to H_m(I;A) \to H_m(I/(g_1);A/(g_1)) \to H_{m-1}(I;A)(-\deg\, g_1) \to 0$$

for all $m \geq 1$; where $H_m(I/(g_1);A/(g_1))$ denotes the Koszul homology of the elements $0,\overline{g}_2,\dots,\overline{g}_r \in A/(g_1)$. From this exact sequence, we have in particular $a(H_m(I;A)) \leq a(H_m(I/(g_1);A/(g_1))$. Denote by "–" the morphism from A to $\overline{A} = A/(g_1,\dots,g_{h-1})$. Repeating $h-1$ times the previous procedure, we get $a(H_m(I;A)) \leq a(H_m(\overline{I};\overline{A}))$ for all $m \geq 1$. But now $\overline{I}$ is a height one ideal in the CM ring $\overline{A}$. Let us denote the Koszul complex of $\overline{I}$ by $\overline{K} = K(\overline{I};\overline{A})$, and the differential from $\overline{K}_{m+1}$ to $\overline{K}_m$ by d_{m+1}. Set $\overline{Z}_m = \operatorname{Ker}(d_m)$, $\overline{B}_m = \operatorname{Im}(d_{m+1})$, $\overline{H}_m = H_m(\overline{I};\overline{A})$. Then there are exact sequences

$$0 \to \overline{B}_m \to \overline{Z}_m \to \overline{H}_m \to 0,$$

$$0 \to \overline{Z}_{m+1} \to \overline{K}_{m+1} \to \overline{B}_m \to 0.$$

By [Hu1, Lemma 1.6] $\overline{H}_m$ are CM modules for all m, and then by [Hu1, Lemma 1.8], $\overline{Z}_m$ and $\overline{B}_m$ are maximal CM modules for $\overline{A}$. The exact sequences imply now $a(\overline{H}_m) \leq a(\overline{B}_m) \leq a(\overline{K}_{m+1})$. Denoting by $\delta_i = \deg(g_i)$,

$$a(\overline{K}_{m+1}) = a(\overline{A}) + \max\{\delta_{i_1}+\ldots+\delta_{i_{m+1}} \mid h \leq i_1 < \ldots < i_{m+1} \leq r\}$$
$$= a(A)+\delta_1+\ldots+\delta_{h-1}+\max\{\delta_{i_1}+\ldots+\delta_{i_{m+1}} \mid h \leq i_1 < \ldots < i_{m+1} \leq r\}$$

$\leq -n + d_1 + \ldots + d_{h+m}.$

So we are done if we prove the following lemma. $\square$

Lemma 5.2.14 *Let $A = k[t_1, \ldots, t_s]$ be a CM graded algebra over an infinite field k, with $\deg(t_i) = 1$. For any homogeneous ideal I, there exists a minimal homogeneous system of generators $g_1, \ldots g_r$ of I such that $g_1, \ldots g_h$ is a maximal regular sequence in I.*

Proof. Set $r = \mu(I)$, $h = \mathrm{ht}\,(I)$. Let $f_1, \ldots, f_r$ be a minimal homogeneous system of generators of I, with $d_i = \deg(f_i)$, $d_1 \leq \ldots \leq d_r = d$. We are going to prove the statement by induction on h. If $h = 0$ there is nothing to prove. Assume $h \geq 1$. Then $I \not\subset z(A) = \bigcup_{\mathfrak{p} \in \mathrm{Ass}(A)} \mathfrak{p}$, and so $I_d \not\subset \mathfrak{p}$, $\forall \mathfrak{p} \in \mathrm{Ass}\,(A)$ (otherwise, we would have $f_i^d \in \mathfrak{p}$ for all i, and then $I \subset \mathfrak{p}$). Since k is infinite, we get $I_d \not\subset \bigcup_{\mathfrak{p} \in \mathrm{Ass}(A)} \mathfrak{p} \cap A_d$, and so there exists $g \in I_d$ such that $g \notin \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}\,(A)$. Note that I_d is a k-vector space generated by the forms $f_{j_1}, \ldots, f_{j_i}$ in degree d and the forms $M f_j$, with $d_j < d$ and M a monomial in $t_1, \ldots, t_s$ of degree $d - d_j$. Now we can write

$$g = \lambda_1 f_{j_1} + \ldots + \lambda_i f_{j_i} + \sum \mu_{jM} M f_j,$$

with $\lambda_1, \ldots, \lambda_i, \mu_{jM} \in k$. If there exists p such that $\lambda_p \neq 0$, we can replace f_{j_p} by g in the minimal system of generators of I. Otherwise, we have an element g in the ideal generated by the forms in I of degree $d' = d_{r-1}$ with the property that $g \notin \mathfrak{p}$, $\forall \mathfrak{p} \in \mathrm{Ass}\,(A)$. We can repeat the arguments for $I_{d'}$, and finally we will replace one of the forms f_j by g. By considering $\overline{A} = A/(g)$, the ideal $\overline{I} = I/(g)$ has $\mu(\overline{I}) = r - 1$, $\mathrm{ht}\,(\overline{I}) = h - 1$. Then we get the result by induction. $\square$

5.2.2 Applications to the study of the diagonals of the Rees algebra

Next we are going to apply the results about the a-invariant of the powers of an ideal to study the Cohen-Macaulay property of the diagonals of the Rees algebra. If the Rees algebra is Cohen-Macaulay, according to Theorem 3.4.13 there exists $\alpha \in \mathbb{Z}$ such that $k[(I^e)_c]$ is Cohen-Macaulay for any $c > de + \alpha$ and $e > 0$. For equigenerated ideals, we obtained $\alpha = d(l-1)$ as upper bound. The following result precises the best α.

Proposition 5.2.15 *Let I be an ideal in $A = k[X_1, \ldots, X_n]$ generated by forms in degree d whose Rees algebra is Cohen-Macaulay. Set $l = l(I)$. For $\alpha \geq 0$, the following are equivalent*

(i) *For all $c > de + \alpha$, $k[(I^e)_c]$ is CM.*

(ii) $a_i(I^e) \leq de + \alpha$, $\forall i$, $\forall e$.

(iii) $a_i(I^e) \leq de + \alpha$, $\forall i$, $\forall e \leq l - 1$.

(iv) $H^{n+1}_{\mathcal{M}}(R_A(I))_{(p,q)} = 0$, $\forall p > dq + \alpha$, *that is,* $\alpha \geq a^1(R^\varphi)$.

(v) *The minimal bigraded free resolution of $R_A(I)$ is good for any diagonal $\Delta = (c, e)$ such that $c > de + \alpha$.*

Proof. If $k[(I^e)_c]$ is CM for $c > de + \alpha$ then we have $H^i_{\mathfrak{m}}(I^e)_c = 0$ for any $i < n$ and $c > de + \alpha$ by Proposition 3.4.1, so $a_*(I^e) \leq de + \alpha$, $\forall e$. The converse follows similarly, and we get the equivalence between (i) and (ii).

Since the Rees algebra R of I is Cohen-Macaulay, we have $a^2_*(R) = a^2(R) = -1$. Then, conditions (ii) and (iii) are equivalent to $a^1(R^\varphi) = a^1_*(R^\varphi) \leq \alpha$ by Theorem 5.2.1.

Finally, we want to prove the equivalence to (v). If the resolution of R is good for diagonals $\Delta = (c, e)$ such that $c > de+\alpha$, then we have $H^i_m(k[(I^e)_c]) = H^{i+1}_{\mathcal{M}}(R)_\Delta = 0$ for $i < n$ by Corollary 2.1.14, so $k[(I^e)_c]$ is Cohen-Macaulay for any $c > de + \alpha$ and we obtain (i). Now assume that $a^1(R^\varphi) \leq \alpha$, and let us consider the minimal bigraded free resolution of R over S

$$0 \to D_t \to \ldots \to D_1 \to D_0 \to R \to 0,$$

with $D_p = \bigoplus_{(a,b)\in\Omega_p} S(a, b)$. By applying the functor $(\ )^\varphi$, we have that

$$0 \to D^\varphi_t \to \ldots \to D^\varphi_0 \to R^\varphi \to 0$$

is the bigraded minimal free resolution of R^φ over S^φ, with $D^\varphi_p = \bigoplus_{(a,b)\in\Omega_p} S^\varphi(a - db, b)$. Therefore, according to Theorem 1.3.4, for any $(a, b) \in \Omega_R$ we have $db - a - n \leq \alpha$. Then the sets X^Δ, Y^Δ introduced in Remark 2.1.11 are empty for diagonals $\Delta = (c, e)$ with $c > de + \alpha$, so the resolution is good for these Δ. $\square$

If the form ring is Gorenstein, we can express this criterion by means of the second a-invariant of the form ring. Namely,

Corollary 5.2.16 *Let I be an ideal in $A = k[X_1, \ldots, X_n]$ generated by forms of degree d whose form ring is Gorenstein. For $\alpha \geq 0$, the following are equivalent*

(i) *For all $c > de + \alpha$, $k[(I^e)_c]$ is CM.*

(ii) $\alpha \geq d(-a^2(G) - 1) - n$.

Proof. By Proposition 5.2.9, $a^1(R^\varphi) = d(-a^2(G) - 1) - n$. Then the result follows from Proposition 5.2.15. □

Let I be an equigenerated ideal in A. If the Rees algebra is Cohen-Macaulay, it can happen that some of its diagonals are not Cohen-Macaulay. Now, by taking $\alpha = 0$ in Proposition 5.2.15 we have a criterion to decide when all the diagonals of a Cohen-Macaulay Rees algebra are Cohen-Macaulay.

Corollary 5.2.17 *Let I be an ideal in $A = k[X_1, \ldots, X_n]$ generated by forms in degree d whose Rees algebra is Cohen-Macaulay. Set $l = l(I)$. Then the following are equivalent*

(i) *For all $c \geq de + 1$, $k[(I^e)_c]$ is CM.*

(ii) $a_i(I^e) \leq de$, $\forall i$, $\forall e \leq l - 1$.

(iii) $H^{n+1}_{\mathcal{M}}(R_A(I))_{(p,q)} = 0, \quad \forall p > dq$.

(iv) *The minimal bigraded free resolution of $R_A(I)$ is good for any Δ.*

Assuming that $G_A(I)$ is Gorenstein, these conditions are also equivalent to

(v) $-a^2(G) \leq \frac{n}{d} + 1$.

Example 5.2.18 We may recover Corollary 3.4.2 as an easy application of Corollary 5.2.17. Let $\{L_{ij}\}$ be a set of $d \times (d+1)$ homogeneous linear forms in a polynomial ring $A = k[X_1, \ldots, X_n]$, and let M be the matrix (L_{ij}). Let $I_t(M)$ be the ideal generated by the $t \times t$ minors of M and assume that $\mathrm{ht}(I_t(M)) \geq d - t + 2$ for $1 \leq t \leq d$. Set $I = I_d(M)$. The ideal I is generated by $d + 1$ forms of degree d, and we have a presentation of the Rees algebra of the form

$$R_A(I) = k[X_1, \ldots, X_n, Y_1, \ldots, Y_{d+1}]/(\phi_1, \ldots, \phi_d),$$

with $\deg(Y_j) = (d, 1)$, $\deg(\phi_i) = (d + 1, 1)$, such that $\phi_1, \ldots, \phi_d$ is a regular sequence. Therefore $R_A(I)$ is Gorenstein, and so $a^2(G_A(I)) = -2$. Since $d \leq n - 1$, we have that $-a^2(G) \leq \frac{n}{d} + 1$. Therefore $k[(I^e)_c]$ is Cohen-Macaulay for any $c \geq de + 1$.

Next, we are going to use the bounds of the a-invariants of the families of ideals considered in Subsection 5.3.1 to study the Cohen-Macaulayness of the diagonals of their Rees algebras. First we recall a well-known result about the vanishing of the graded pieces of the local cohomology modules.

Lemma 5.2.19 *Let A be a standard noetherian graded k-algebra with graded maximal ideal $\mathfrak{m}$. Let L be a finitely generated graded A-module with $d = \dim L > 0$. Then*

$$H^d_{\mathfrak{m}}(L)_j \neq 0,\ \forall j \leq a(L).$$

Proof. Since $d > 0$, we can assume $H^0_{\mathfrak{m}}(L) = 0$ because otherwise by considering $\overline{L} = L/H^0_{\mathfrak{m}}(L)$ we have $H^0_{\mathfrak{m}}(\overline{L}) = 0$ and $H^d_{\mathfrak{m}}(\overline{L}) = H^d_{\mathfrak{m}}(L)$. We may also assume that the field k is infinite. Then there exists $x \in A_1$ such that $x \notin z_A(L)$, and the exact sequence

$$0 \to L(-1) \xrightarrow{\cdot x} L \to L/xL \to 0$$

induces the graded exact sequence of local cohomology modules

$$H^{d-1}_{\mathfrak{m}}(L/xL) \to H^d_{\mathfrak{m}}(L)(-1) \to H^d_{\mathfrak{m}}(L) \to 0.$$

From this exact sequence, we have that $H^d_{\mathfrak{m}}(L)_s = 0$ implies $H^d_{\mathfrak{m}}(L)_j = 0$ for $j \geq s$, so we are done. $\square$

Proposition 5.2.20 *Let I be an equimultiple ideal in A of height $h > 1$ generated by forms in degree d whose Rees algebra is Cohen-Macaulay. For any $c \geq de + 1$, $k[(I^e)_c]$ is Cohen-Macaulay if and only if $c > d(e-1) + a(A/I)$.*

Proof. We have proved in Proposition 5.2.8 that $a^1(R^{\varphi}) = a(A/I) - d$. Therefore, $k[(I^e)_c]$ is Cohen-Macaulay for any $c > d(e-1) + a(A/I)$ by Proposition 5.2.15.

On the other hand, since $a_{n-h}(I^e/I^{e+1}) = de + a(A/I)$ by Proposition 5.2.8, we have $H^{n-h}_{\mathfrak{m}}(I^e/I^{e+1})_s \neq 0$ for all $s \leq de + a(A/I)$ according to Lemma 5.2.19. By considering the short exact sequences

$$0 \to I^{e+1} \to I^e \to I^e/I^{e+1} \to 0,$$

and by induction on e, we get $H^{n-h+1}_{\mathfrak{m}}(I^e)_s \neq 0$ for all $s \leq d(e-1) + a(A/I)$. Now, if $k[(I^e)_c]$ is Cohen-Macaulay then $H^i_{\mathfrak{m}}(I^{es})_{cs} = 0$ for $i < n$ and $s > 0$ by Proposition 3.4.1. In particular, it holds $H^{n-h+1}_{\mathfrak{m}}(I^e)_c = 0$, and so $c > d(e-1) + a(A/I)$. This proves the converse. $\square$

Proposition 5.2.21 *Let I be a strongly Cohen-Macaulay ideal such that $\mu(I_{\mathfrak{p}}) \leq \mathrm{ht}\,(\mathfrak{p})$ for any prime ideal $\mathfrak{p} \supseteq I$. Assume that I is minimally generated by r forms of degree $d = d_1 \geq \ldots \geq d_r$, and let $h = \mathrm{ht}\,(I)$. For $c > d(e-1) + d_1 + \ldots + d_h - n$, $k[(I^e)_c]$ is Cohen-Macaulay.*

Proof. According to Corollary 3.4.4, for a given $c \geq de + 1$ we have that $k[(I^e)_c]$ is Cohen-Macaulay if and only if $H^i_{\mathfrak{m}}(I^{es})_{cs} = 0$, for $i < n$, $s > 0$, and $H^i_{\mathfrak{m}}(I^{es-h+1})_{cs-n} = 0$, for $1 < i \leq n$, $s > 0$.

From Proposition 5.2.13, note that $a_*(I^e) \leq (e-1)d + d_1 + \ldots + d_h - n$. Therefore, to get the vanishing of the cohomology modules it suffices to see that $cs > (es-1)d + d_1 + \ldots + d_h - n$ and $cs - n > (es-h)d + d_1 + \ldots + d_h - n$ for any $s \geq 1$. The first condition is equivalent to $(c - de)s > d_1 + \ldots + d_h - d - n$ for $s \geq 1$, that is, $c - de > d_1 + \ldots + d_h - d - n$. The second one is equivalent to $(c - de)s > d_1 + \ldots + d_h - dh$ for $s \geq 1$, that is, $c - de > d_1 + \ldots + d_h - dh$; and this always holds because $d_1 + \ldots + d_h - dh \leq 0$.□

To finish, let us consider the case that the Rees algebra has rational singularities. Then all the diagonals $k[(I^e)_c]$ have rational singularities by [Bou], so in particular the Rees algebra and its diagonals are Cohen-Macaulay. By Proposition 3.4.1, we get immediately

Proposition 5.2.22 *Let I be a homogeneous ideal in $A = k[X_1, \ldots, X_n]$ generated by forms of degree $\leq d$, where k is a field with $\mathrm{char}\, k = 0$. If $R_A(I)$ has rational singularities, then $a_*(I^e) \leq de$ for all e.*

Example 5.2.23 Let $\mathbf{X} = (X_{ij})$ be an $m \times n$ matrix of indeterminates, with $1 \leq i \leq m$, $1 \leq j \leq n$ and $m \leq n$. Let $A = k[\mathbf{X}]$ be the polynomial ring with variables the entries in $\mathbf{X}$, where k is a field with char $k = 0$ and $k = \overline{k}$. Let $I = I_d(\mathbf{X})$ be the ideal generated by the d-minors of $\mathbf{X}$, $1 < d < m$.

By [Bru, Theorem 3.2], $R_A(I)$ has rational singularities. So we have $a_*(I^e) \leq de$, for all e. This also holds for ideals generated by minors of symmetric generic matrices and ideals generated by pfaffians of alternating generic matrices by [Bru, Remark 3.4]. The defining ideals of the varieties considered by A. Bertram in [Ber] are:

(a) $I_2(\mathbf{X})$, with $\mathbf{X}$ a generic matrix, for the defining ideals of the products $\mathbb{P}^r_k \times \mathbb{P}^s_k$.

(b) $I_2(\mathbf{X})$, with $\mathbf{X}$ a generic symmetric matrix, for the defining ideals of quadratic Veronese embeddings of $\mathbb{P}^r_k$.

(c) $Pf_2(\mathbf{X})$, the ideal generated by the pfaffians of a generic alternating matrix, for the defining ideal of the Plücker embedding of $G(2,r)$.

In these cases we get $a_*(I^e) \leq 2e$, for all e. A. Bertram gets the following bounds for $M = \max_{i\geq 2}\{a_i(I^e)\}$:

(a) $M \leq 2e - 4$.

(b) $M \leq 2e - 3$.

(c) $M \leq 2e - 6$.

5.3 Bayer–Stillman Theorem

Let $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ be the polynomial ring in $n + r$ variables with the bigrading given by $\deg(X_i) = (1, 0)$, $\deg(Y_j) = (0, 1)$, so that S is a standard bigraded k-algebra. For a homogeneous ideal I in S, we have already defined in Section 5.1 the bigraded regularity $\mathbf{reg}(I)$ of I. The aim of this section is to give a new description of the regularity of I analogous to the one given by D. Bayer and M. Stillman [BaSt] in the graded case. To this end, we are going to prove several technical lemmas which are the bigraded version of the ones in [BaSt]. To state them, we need to introduce the *saturations* of I with respect to the variables $\underline{X}$ and $\underline{Y}$ (I^{*1} and I^{*2}), and the generic forms for I with respect to $\underline{X}$ and $\underline{Y}$. Furthermore, Theorem 5.1.1 and Theorem 5.1.2 will be needed to prove some of these lemmas. We will include all the proofs for the completeness.

For a given finitely generated bigraded S-module L, we say that L is $(m, \cdot)$-regular if $\operatorname{reg}_1 L \leq m$. Similarly, L is $(\cdot, m)$-regular if $\operatorname{reg}_2 L \leq m$. Denote by $\mathcal{M}_1$ and $\mathcal{M}_2$ the ideals of S generated by $\mathfrak{m}_1 = (X_1, ..., X_n)$ and $\mathfrak{m}_2 = (Y_1, \ldots, Y_r)$ respectively. Then we have

Proposition 5.3.1 *Let L be a finitely generated bigraded S-module. Then the following are equivalent:*

(i) L is $(m, \cdot)$-regular.

(ii) $H^i_{\mathcal{M}_1}(L)_{(p,q)} = 0$ for all i, q, $p \geq m - i + 1$.

Proof. By Theorem 5.1.2, L is $(m,\cdot)$-regular if and only if $\operatorname{reg}(L^q) \leq m$ for any q, that is, $H^i_{\mathfrak{m}_1}(L^q)_p = 0$ for all i, q and $p \geq m - i + 1$. Now the result follows from Proposition 2.1.18. □

Given a homogeneous ideal I, let us define I^{*1} and I^{*2} to be the homogeneous ideals

$$I^{*1} = \{f \in S : \exists\, k \; s.t.\; {\mathcal{M}_1}^k f \subset I\},$$
$$I^{*2} = \{f \in S : \exists\, k \; s.t.\; {\mathcal{M}_2}^k f \subset I\}.$$

Lemma 5.3.2 *Assume k infinite, and let $s = \max\{i \mid H^i_{\mathcal{M}_1}(S/I) \neq 0\}$. Then,*

(i) If $s = 0$, $I^{*1} = S$.

(ii) If $s > 0$, there exists $h \in S_{(1,0)}$ such that $(I^{*1} : h) = I^{*1}$.

Proof. First note that

$$\begin{aligned} H^0_{\mathcal{M}_1}(S/I) &= \{\,\overline{f} \in S/I \mid \exists\, k \; s.t.\; {\mathcal{M}_1}^k\,\overline{f} = 0\} = \\ &= \{\, f \in S \mid \exists\, k \; s.t.\; {\mathcal{M}_1}^k\, f \subset I\}/I = I^{*1}/I. \end{aligned}$$

If $s = 0$, we have $H^i_{\mathfrak{m}_1}((S/I)^q)_p = H^i_{\mathcal{M}_1}(S/I)_{(p,q)} = 0$ for any p, q, $i > 0$, so $(S/I)^q$ has dimension 0 as graded S_1-module, and then $H^0_{\mathfrak{m}_1}((S/I)^q) = (S/I)^q$. Therefore, $H^0_{\mathcal{M}_1}(S/I) = S/I$, and we get $I^{*1} = S$.

If $s > 0$, note that $I^{*1} \neq S$ because otherwise $H^0_{\mathcal{M}_1}(S/I) = S/I$ and then we would have $H^i_{\mathcal{M}_1}(S/I) = 0$ for all $i > 0$. Now consider $\overline{S} = S/I^{*1}$, and denote by $T = \overline{S}^0 = S_1/(I^{*1})^0$, $\overline{\mathfrak{m}_1} = \mathfrak{m}_1 T$. We have that $H^0_{\mathcal{M}_1}(\overline{S}) = (I^{*1})^{*1}/I^{*1} = 0$, and so $H^0_{\mathfrak{m}_1}(\overline{S}^q) = 0$ for all q. Therefore, $\overline{\mathfrak{m}_1} \notin \bigcup_q \operatorname{Ass}_T(\overline{S}^q)$. On the other hand, according to [HIO, Proposition 23.6] we have that $\bigcup_q \operatorname{Ass}_T(\overline{S}^q)$ is a finite set. Since k is infinite, we can find $h \in S_1$ of degree 1 such that $\overline{h} \notin z_T(\overline{S}^q)$ for all q. Then $h \in S_{(1,0)}$ satisfies that $h \notin z_{S_1}(S/I^{*1})$. Therefore, $(I^{*1} : h) = I^{*1}$. □

From now on in this section we will assume that the field k is infinite. Let $s = \max\{i \mid H^i_{\mathcal{M}_1}(S/I) \neq 0\}$. If $s > 0$, $h \in S_{(1,0)}$ is generic for I if $h \notin z_{S_1}(S/I^{*1})$, that is, $(I^{*1} : h) = I^{*1}$. If $s = 0$, we say that any $h \in S_{(1,0)}$ is generic for I. Given $j > 0$, we define $U^1_j(I)$ to be the set

$$\{(h_1, \ldots, h_j) \in S^j_{(1,0)} \mid h_i \text{ is generic for } (I, h_1, \ldots, h_{i-1}), 1 \leq i \leq j\}.$$

Lemma 5.3.3 *Let $h \in S_{(1,0)}$. The following are equivalent:*

(i) $(I : h)_{(p,q)} = I_{(p,q)}$ for $p \geq m$.

(ii) h is generic for I and $(I^{*1})_{(p,q)} = I_{(p,q)}$ for $p \geq m$.

Proof. First, let us notice that for $p > a^1_*(S/I)$ we have

$$(I^{*1}/I)_{(p,q)} = H^0_{\mathcal{M}_1}(S/I)_{(p,q)} = H^0_{\mathfrak{m}_1}((S/I)^q)_p = 0$$

by Theorem 5.1.1. Therefore, for p large enough it holds $(I^{*1})_{(p,q)} = I_{(p,q)}$, $\forall q$.

Now let us assume that (i) holds, and let $f \in I^{*1}$ be a homogeneous element not in I such that $\deg_1 f$ is maximum. Then $hf \in I^{*1}$ has $\deg_1(hf) > \deg_1 f$, so $hf \in I$. Hence $\deg_1 f < m$, and $(I^{*1})_{(p,q)} = I_{(p,q)}$ for any $p \geq m$. To show that h is generic for I, we may assume that $s = \max\{i \mid H^i_{\mathcal{M}_1}(S/I) \neq 0\} > 0$ (if not, any element in $S_{(1,0)}$ is generic for I). Then we want to prove $h \notin z_{S_1}(S/I^{*1})$. Otherwise, there exists a homogeneous element $f \notin I^{*1}$ such that $hf \in I^{*1}$. By Lemma 5.3.2, there exists $g \in S_{(1,0)}$ such that $g \notin z_{S_1}(S/I^{*1})$. Then, for any $s \geq 0$ we have $g^s f \notin I^{*1}$ and $hg^s f \in I^{*1}$, so $(I^{*1} : h)_{(p,q)} \neq (I^{*1})_{(p,q)}$ for all $p \gg 0$. But note that for any $p \geq m$,

$$(I^{*1} : h)_{(p,q)} = (I : h)_{(p,q)} = I_{(p,q)} = (I^{*1})_{(p,q)},$$

and we get a contradiction.

Now assuming (ii), we have that for $p \geq m$

$$(I : h)_{(p,q)} = (I^{*1} : h)_{(p,q)} = (I^{*1})_{(p,q)} = I_{(p,q)}. \ \square$$

Lemma 5.3.4 *Let $h \in S_{(1,0)}$ be generic for I. The following are equivalent:*

(i) I is $(m, \cdot)$-regular.

(ii) (I, h) is $(m, \cdot)$-regular and $(I^{*1})_{(p,q)} = I_{(p,q)}$ for all $p \geq m$.

Proof. If I is $(m, \cdot)$-regular, then S/I is $(m-1, \cdot)$-regular. Then for any i, q and $p \geq m - i$, we have $H^i_{\mathcal{M}_1}(S/I)_{(p,q)} = 0$ by Proposition 5.3.1. In particular, for $p \geq m$

$$0 = H^0_{\mathcal{M}_1}(S/I)_{(p,q)} = (I^{*1}/I)_{(p,q)},$$

and so $(I^{*1})_{(p,q)} = I_{(p,q)}$ for $p \geq m$.

Let us consider $Q := (I : h)/I$. In the assumptions of (i) or (ii), observe that for $p \geq m$, $(I : h)_{(p,q)} = (I^{*1} : h)_{(p,q)} = (I^{*1})_{(p,q)} = I_{(p,q)}$, so $Q_{(p,q)} = 0$ for $p \geq m$. Therefore $H^i_{\mathcal{M}_1}(Q) = 0$ for all $i > 0$ and $H^0_{\mathcal{M}_1}(Q) = Q$. From the bigraded exact sequence

$$0 \to I \to (I : h) \to Q \to 0,$$

the long exact sequence of local cohomology gives

$$H^i_{\mathcal{M}_1}(I)_{(p,q)} \cong H^i_{\mathcal{M}_1}((I:h))_{(p,q)}\ , \forall i, p \geq m-i+1.$$

Assume first (i). We have already shown that $(I^{*1})_{(p,q)} = I_{(p,q)}$ for all $p \geq m$. Since I is $(m,\cdot)$-regular, we have $H^i_{\mathcal{M}_1}((I:h))_{(p,q)} = 0$ for all i, $p \geq m-i+1$. By considering the exact sequence

$$0 \to I \cap (h) = (I:h)h = (I:h)(-1,0) \to I \oplus (h) \to (I,h) \to 0,$$

we get $H^i_{\mathcal{M}_1}((I,h))_{(p,q)} = 0$ for all i, $p \geq m-i+1$, so (I,h) is $(m,\cdot)$-regular.

Now by assuming (ii), from the previous exact sequence we obtain that $H^i_{\mathcal{M}_1}((I:h))_{(p-1,q)} \cong H^i_{\mathcal{M}_1}(I)_{(p,q)}$ for $p \geq m-i+2$. For $p \geq m-i+1$, we then have that $H^i_{\mathcal{M}_1}(I)_{(p,q)} \cong H^i_{\mathcal{M}_1}((I:h))_{(p,q)} \cong H^i_{\mathcal{M}_1}(I)_{(p+1,q)}$. Therefore $H^i_{\mathcal{M}_1}(I)_{(p,q)} = 0$ for $p \geq m-i+1$, so I is $(m,\cdot)$-regular. $\square$

Lemma 5.3.5 *Let I be an ideal generated by forms in* $\deg_1 \leq m$ *and* $h \in S_{(1,0)}$. *If (I,h) is $(m,\cdot)$-regular, then $(I:h)$ is generated by forms in* $\deg_1 \leq m$.

Proof. Let $f_1, \ldots, f_u, hf_{u+1}, \ldots, hf_v$ be a minimal system of homogeneous generators for I, where $f_1, \ldots, f_u, h$ is a minimal system of generators for (I,h). If $f \in (I:h)$, then

$$hf = g_1 f_1 + \ldots + g_u f_u + h(g_{u+1} f_{u+1} + \ldots + g_v f_v),$$

for $g_1, \ldots, g_v \in S$. Thus

$$(f - g_{u+1} f_{u+1} - \ldots - g_v f_v)h - g_1 f_1 - \ldots - g_u f_u = 0.$$

The first map in the bigraded minimal free resolution of (I,h) is

$$\begin{array}{ccc} Se \oplus Se_1 \oplus \ldots \oplus Se_u & \longrightarrow & (I,h) \\ e & \longmapsto & h \\ e_j & \longmapsto & f_j \end{array}$$

and we have that

$$(f - g_{u+1} f_{u+1} - \ldots - g_v f_v)e - g_1 e_1 - \ldots - g_u e_u$$

is a first syzygy of (I,h). Conversely, if $le + l_1 e_1 + \ldots + l_u e_u$ is a first syzygy of (I,h) then $lh + l_1 f_1 + \ldots + l_u f_u = 0$, so $lh \in (f_1, \ldots, f_u) \subset I$, and $l \in (I:h)$.

Because (I,h) is $(m,\cdot)$-regular, each first syzygy of (I,h) can be expressed in terms of syzygies of (I,h) in $\deg_1 \leq m+1$. Then

$$(f-g_{u+1}f_{u+1}-\ldots-g_vf_v)e-g_1e_1-\ldots-g_ue_u=\sum_i \lambda_i(\gamma_ie+\gamma_{i1}e_1+\ldots+\gamma_{iu}e_u),$$

with $\deg_1(\gamma_ie+\gamma_{i1}e_1+\ldots+\gamma_{iu}e_u)\leq m+1$. So

$$f=g_{u+1}f_{u+1}+\ldots+g_vf_v+\sum_i \lambda_i\gamma_i,$$

with $\gamma_i \in (I:h)$, $\deg_1 \gamma_i \leq m$. Since $f_{u+1},\ldots,f_v$ also belong to $(I:h)$ and have $\deg_1 \leq m$, we finally obtain that $(I:h)$ can be generated by elements in $\deg_1 \leq m$. $\square$

We are now ready to prove a bigraded version of the Bayer-Stillman's Theorem characterizing the regularity of a homogeneous ideal in terms of generic forms.

Theorem 5.3.6 *Let I be a homogeneous ideal in S generated by forms in $\deg_1 \leq m$. Then the following are equivalent:*

(i) I is $(m,\cdot)$-regular.

(ii) There exist $h_1,\ldots,h_j \in S_{(1,0)}$ for some $j\geq 0$ such that

$$((I,h_1,\ldots,h_{i-1}):h_i)_{(m,q)}=(I,h_1,\ldots,h_{i-1})_{(m,q)},\ \forall q,\ 1\leq i\leq j.$$

$$(I,h_1,\ldots,h_j)_{(m,q)}=S_{(m,q)},\ \forall q.$$

(iii) Let $s=\max\{i \mid H^i_{\mathcal{M}_1}(S/I)\neq 0\}$. For all $(h_1,\ldots,h_s)\in U^1_s(I)$, $p\geq m$,

$$((I,h_1,\ldots,h_{i-1}):h_i)_{(p,q)}=(I,h_1,\ldots,h_{i-1})_{(p,q)},\ \forall q,\ 1\leq i\leq s.$$

$$(I,h_1,\ldots,h_s)_{(p,q)}=S_{(p,q)},\ \forall q.$$

Proof. Note that $(iii)\Rightarrow(ii)$ is obvious. Now we are going to show that $(ii)\Rightarrow(i)$ by induction on j. If $j=0$, we have that $I_{(m,q)}=S_{(m,q)}$ for all q, so $I_{(p,q)}=S_{(p,q)}$ for all q and $p\geq m$. Therefore,

$$H^i_{\mathcal{M}_1}(S/I)=\begin{cases}0 & \text{if } i>0\\ S/I & \text{if } i=0\end{cases}.$$

In particular, we have that $H^i_{\mathcal{M}_1}(S/I)_{(p,q)}=0$ for all i, q and $p\geq m-i$, so I is $(m,\cdot)$-regular. If $j>0$, we have that (I,h_1) is $(m,\cdot)$-regular by the induction

hypothesis. Since I is generated by forms in $\deg_1 \le m$, we have that $(I : h_1)$ is generated by forms in $\deg_1 \le m$ by Lemma 5.3.5. As $(I : h_1)_{(m,q)} = I_{(m,q)}$, we then conclude $(I : h_1)_{(p,q)} = I_{(p,q)}$ for all $p \ge m$. According to Lemma 5.3.3, we have that h_1 is generic for I and $(I^{*1})_{(p,q)} = I_{(p,q)}$ for all $p \ge m$. Then I is $(m, \cdot)$-regular by Lemma 5.3.4.

Now let us prove $(i) \Rightarrow (iii)$ by induction on s. If $s = 0$, since I is $(m, \cdot)$-regular we have $H^0_{\mathcal{M}_1}(S/I)_{(p,q)} = (I^{*1}/I)_{(p,q)} = 0$ for $p \ge m$, and $I^{*1} = S$ by Lemma 5.3.2. Therefore, $I_{(p,q)} = (I^{*1})_{(p,q)} = S_{(p,q)}$ for $p \ge m$. Assume now $s > 0$. Since I is $(m, \cdot)$-regular and h_1 is generic for I, we get that (I, h_1) is $(m, \cdot)$-regular and $(I^{*1})_{(p,q)} = I_{(p,q)}$ for all $p \ge m$ by Lemma 5.3.4. As $(h_2, \ldots, h_s) \in U^1_{s-1}((I, h_1))$, by the induction assumption it is just enough to show $(I : h_1)_{(p,q)} = I_{(p,q)}$ for $p \ge m$, which is satisfied by Lemma 5.3.3. $\square$

We are going to use this criterion to compute the bigraded regularity of the generic initial ideal of a homogeneous ideal I in S. Let $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ be the polynomial ring over an infinite field k with the bigrading given by $\deg(X_i) = (1, 0)$, $\deg(Y_j) = (0, 1)$. Let $<$ be an order on the monomials of S. Let us denote by $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$, with $\mathcal{G}_1 = GL(n, k)$, $\mathcal{G}_2 = GL(r, k)$. Given an element $g = (f, h) \in \mathcal{G}$, where $f = (f_{ij})_{1 \le i,j \le n}$ and $h = (h_{ij})_{1 \le i,j \le r}$, g acts on S by acting on the variables in the following way

$$X_j \longmapsto \sum_{i=1}^{n} f_{ij} X_i \quad , \quad Y_j \longmapsto \sum_{i=1}^{r} h_{ij} Y_i.$$

We will denote by $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2$, where $\mathcal{B}_1, \mathcal{B}_2$ are the Borel subgroups of $\mathcal{G}_1, \mathcal{G}_2$ consisting of upper triangular matrices, and by $\mathcal{B}' = \mathcal{B}'_1 \times \mathcal{B}'_2$, where $\mathcal{B}'_1, \mathcal{B}'_2$ are the Borel subgroups of $\mathcal{G}_1, \mathcal{G}_2$ consisting of lower triangular matrices. We will denote by $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2$, where $\mathcal{U}_1, \mathcal{U}_2$ are the unipotent matrices. By bigrading the proof of [Eis, Theorem 15.18], we get

Theorem 5.3.7 *(Galligo, Bayer–Stillman) Let $I \subset S$ be a homogeneous ideal. There exists a non-empty Zariski open $U = \mathcal{B}'U \subset \mathcal{G}$, $U \cap \mathcal{U} \neq Id$, and a monomial ideal J such that*

$$\mathrm{in}(gI) = J, \ \forall g \in U.$$

We call J the (bi)graded generic initial ideal of I, written $J = \mathbf{gin}(I)$. Given a homogeneous ideal $I \subset S$, we say that I is Borel-fix if $gI = I$ for any $g \in \mathcal{B}$. It was proved that the generic initial ideal of a graded ideal is Borel

fix. By bigrading the proof of [Eis, Theorem 15.20], we easily obtain that the generic initial ideal is Borel-fix.

Theorem 5.3.8 *Let $I \subset S$ be a homogeneous ideal. For any $g \in \mathcal{B}$,*

$$g(\mathbf{gin}(I)) = \mathbf{gin}(I).$$

Let $p \geq 0$. Given $s, t \in \mathbb{N}$, we define $s <_p t \iff \binom{t}{s} \not\equiv 0 \ (mod\, p)$. We also can give an equivalent characterization of the Borel-fix bihomogeneous ideals analogous to the one in the graded case [Eis, Theorem 15.23]. Namely,

Theorem 5.3.9 *Let I be a homogeneous ideal of S. Let $p = char k \geq 0$. Then*

(i) I is diagonal-fix iff I is monomial.

(ii) I is Borel-fix iff I is generated by monomials m such that satisfy the following conditions

- *If m is divisible by X_j^t but by no higher power of X_j, then $(X_i/X_j)^s m \in I$, $\forall i < j$, $s <_p t$.*
- *If m is divisible by Y_j^t but by no higher power of Y_j, then $(Y_i/Y_j)^s m \in I$, $\forall i < j$, $s <_p t$.*

For a homogeneous ideal I, let us denote by $\delta_1(I)$ the maximum first component of the degree in a minimal system of generators of I. In a similar way, we may define $\delta_2(I)$. Then we have

Proposition 5.3.10 *Let $I \subset S$ be a Borel-fix ideal. If $char k = 0$, then*

$$\operatorname{reg}_1(I) = \delta_1(I),$$

$$\operatorname{reg}_2(I) = \delta_2(I).$$

Proof. Set $m = \delta_1(I)$. From the definition of the regularity it is clear that $\operatorname{reg}_1(I) \geq m$. According to Theorem 5.3.6, to prove the equality it is enough to show that for $p \geq m$, $i \leq n$, we have

$$((I, X_n, \ldots, X_{i+1}) : X_i)_{(p,q)} = (I, X_n, \ldots, X_{i+1})_{(p,q)}.$$

Let $f \in ((I, X_n, \dots, X_{i+1}) : X_i)$ be a monomial with $\deg_1 f \geq m$. If there exists $k \geq i+1$ such that $X_k | f$, we immediately have $f \in (I, X_n, \dots, X_{i+1})$. Otherwise,

$$X_i f = X^\alpha Y^\beta (X^A Y^B),$$

where $X^A Y^B \in I$, $\deg_1 X^A \leq m$, $\deg_1 X^\alpha \geq 1$. If $X_i | X^\alpha$, we then easily get $f \in I$. If not, by taking $k \leq i$ such that $X_k | X^\alpha$, we can write

$$f = \frac{X^\alpha}{X_k} Y^\beta (\frac{X_k}{X_i} X^A Y^B).$$

Since I is Borel-fix, we have that $\frac{X_k}{X_i} X^A Y^B \in I$ by Theorem 5.3.9, and so $f \in I \subset (I, X_n, \dots, X_{i+1})$. $\square$

This result has been proved independently by A. Aramova et al. [ACD] by other methods.

In the graded case, it was proved by D. Bayer and M. Stillman [BaSt] that there exists an order in $A = k[X_1, \dots, X_n]$ (the reverse lexicographic order) with the property that $\operatorname{reg}(I) = \operatorname{reg}(\operatorname{gin} I)$ for any homogeneous ideal I in A. We may wonder if the analogous bigraded result also holds, that is, if there exists an order in the polynomial ring S such that $\mathbf{reg}(I) = \mathbf{reg}(\mathbf{gin}\, I)$ for any homogeneous ideal I. We show that this is not true by giving a homogeneous ideal in S such that $\mathbf{reg}(I) \neq \mathbf{reg}(\mathbf{gin}\, I)$ for any order on S.

Example 5.3.11 Let us consider the polynomial ring $S = k[X_1, X_2, Y_1, Y_2]$, with $\deg(X_1) = \deg(X_2) = (1,0)$, $\deg(Y_1) = \deg(Y_2) = (0,1)$. Let $>$ be a term order in S, that is, an order satisfying

(i) $X_1 > X_2$, $Y_1 > Y_2$.

(ii) For monomials m, m_1, m_2 in S, if $m_1 > m_2$ then $mm_1 > mm_2$.

Let I be the homogeneous ideal in S generated by the forms $f_1 = X_1 Y_1$ and $f_2 = X_1 Y_2 + X_2 Y_1$ in degree $(1,1)$. Note that f_1, f_2 is a regular sequence, so the Koszul complex of these forms provides the minimal bigraded free resolution of I:

$$0 \to S(-2,-2) \to S(-1,-1)^2 \to I \to 0.$$

Then the regularity of I is $\mathbf{reg}(I) = (1,1)$. Note that $X_1 Y_1 > X_1 Y_2$, $X_2 Y_1 > X_2 Y_2$. Therefore, if we want to define an order on the monomials of S we only

must decide if $X_1Y_2 > X_2Y_1$ or $X_1Y_2 < X_2Y_1$ in degree $(1,1)$. Assume first that $X_1Y_2 > X_2Y_1$. Recall that $g \in GL(2,k) \times GL(2,k)$, with

$$g = (A, B) = \left(\begin{pmatrix} a & c \\ b & d \end{pmatrix}, \begin{pmatrix} \alpha & \gamma \\ \beta & \rho \end{pmatrix} \right),$$

operates in S by means of

$$\begin{array}{ccc} X_1 & \longmapsto & aX_1 + bX_2 \\ X_2 & \longmapsto & cX_1 + dX_2 \\ Y_1 & \longmapsto & \alpha Y_1 + \beta Y_2 \\ Y_2 & \longmapsto & \gamma Y_1 + \rho Y_2 \end{array}$$

Since $\dim_k \mathbf{gin}(I)_{(i,j)} = \dim_k I_{(i,j)}$ for any (i,j), we have that $\mathbf{gin}(I)_{(i,j)} = 0$ for $(i,j) \in \{(0,0),(1,0),(0,1)\}$. In degree $(1,1)$, the forms f_1, f_2 are a k-basis of $I_{(1,1)}$. By computing $g(f_1 \wedge f_2)$, we get

$$g(f_1 \wedge f_2) = a^2(\alpha\rho - \beta\gamma)X_1Y_1 \wedge X_1Y_2 + \ldots$$

so $\mathbf{gin}(I)_{(1,1)}$ is the k-vector space generated by X_1Y_1, X_1Y_2. If $\mathbf{gin}(I) = (X_1Y_1, X_1Y_2)$, then $\dim_k \mathbf{gin}(I)_{(1,2)} = 3$ because $X_1Y_1^2$, $X_1Y_1Y_2$, $X_1Y_2^2$ is a k-basis, which is a contradiction because $\dim_k I_{(1,2)} \geq 4$. Therefore, $\mathbf{gin}(I)$ has minimal generators with $\deg_1 \geq 2$ or $\deg_2 \geq 2$, so $\operatorname{reg}_1(\mathbf{gin}\, I) \geq 2$ or $\operatorname{reg}_2(\mathbf{gin}\, I) \geq 2$. In the case $X_1Y_2 < X_2Y_1$, it can be proved that $\mathbf{reg}(\mathbf{gin}\, I) \neq (1,1)$ by similar arguments. Therefore, we get $\mathbf{reg}(I) \neq \mathbf{reg}(\mathbf{gin}\, I)$ for any order in S.

Finally, note that these results can be applied to study the Koszul property of the diagonals of a bigraded standard k-algebra. By using [ERT, Theorem 18], and following the same lines as [ERT, Theorem 2] in the graded case, it can be proved that that for a homogeneous ideal I of S, $(S/I)_\Delta$ has a Gröbner basis of quadrics for $c \gg 0$, $e \gg 0$ (see also [ACD]).

Chapter 6

Asymptotic behaviour of the powers of an ideal

Let $A = k[X_1, \ldots, X_n]$ be a polynomial ring over a field k, and let I be a homogeneous ideal in A. In this chapter we are concerned with the asymptotic behaviour of the powers of I. We will use the bigraded structure of the Rees algebra to get information about the Hilbert polynomials, the Hilbert series and the graded minimal free resolutions of the powers of I.

In Section 6.1 we show that the Hilbert polynomials of the powers of the ideal I have a uniform behaviour. In particular, the Hilbert polynomials of a finite set of these powers allow to compute the Hilbert polynomials of its Rees algebra and its form ring, without needing an explicit presentation of these algebras. In Section 6.2, similar results are stated for the Hilbert series of the powers of I.

The last section begins by studying the projective dimension of the powers of I. The approach to this question by means of the bigrading of the Rees algebra allows to recover some classical results as the constant asymptotic value for the projective dimension, as well as to determine the powers of the ideal which take the asymptotic value whenever the form ring is Gorenstein. After that, we study the graded minimal free resolutions of the powers of an ideal. In the equigenerated case, it is proved that the shifts are given by linear functions asymptotically and the graded Betti numbers of these resolutions are given by polynomials asymptotically. This result is then applied to guess the resolutions of the powers of some families of ideals from a finite set of these resolutions.

6.1 Hilbert polynomial of the powers of an ideal

First of all, let us recall some standard definitions and notations referred to the Hilbert polynomial (see for instance [BH1]). Let $A = k[X_1, \dots, X_n]$ be the polynomial ring over a field k, and let M be a finitely generated graded A-module. The numerical function

$$\begin{aligned} H(M,\ ) : \mathbb{Z} &\longrightarrow \mathbb{Z} \\ j &\mapsto \dim_k M_j \end{aligned}$$

is the Hilbert function of M. Denoting by $d = \dim M$, there exists an unique polynomial $P_M(s) \in \mathbb{Q}[s]$, of degree $d-1$, for which $H(M,j) = P_M(j)$ for all $j \gg 0$. We can write

$$P_M(s) = \sum_{k=0}^{d-1} (-1)^{d-1-k} e_{d-1-k} \binom{s+k}{k},$$

with $e_0, \dots, e_{d-1} \in \mathbb{Z}$. $P_M(s)$ is called the Hilbert polynomial of M.

Our first result shows the uniform behaviour of the Hilbert polynomial of the powers of any homogeneous ideal in a polynomial ring.

Theorem 6.1.1 *Let I be a homogeneous ideal in A. Set $c = a_*^2(R_A(I))$, $h = \mathrm{ht}\,(I)$. Then there are polynomials $e_0(j), \dots, e_{n-h-1}(j)$ with integer values such that for all $j \geq c+1$*

$$P_{A/I^j}(s) = \sum_{k=0}^{n-h-1} (-1)^{n-h-1-k} e_{n-h-1-k}(j) \binom{s+k}{k}.$$

Furthermore, $\deg e_{n-h-1-k}(j) \leq n-k-1$ *for all k.*

Proof. Assume that I is generated by forms $f_1, \dots, f_r$ in degrees $d_1 \leq \dots \leq d_r = d$ respectively. Then the Rees algebra $R = R_A(I)$ of I can be endowed with the bigrading given by $R_{(i,j)} = (I^j)_i$, so that R is a bigraded S-module, for $S = k[X_1, \dots, X_n, Y_1, \dots, Y_r]$ the polynomial ring with $\deg(X_i) = (1,0)$, $\deg(Y_j) = (d_j, 1)$. Since R is a domain, it has relevant dimension $n+1$. Then by Proposition 1.5.1 and Proposition 1.5.5 there exists a polynomial $P_R(s,t)$ of total degree $n-1$ such that for all (i,j)

$$\dim_k R_{(i,j)} - P_R(i,j) = \sum_q (-1)^q \dim_k H^q_{R_+}(R)_{(i,j)},$$

where R_+ is the ideal generated by the products $X_i f_j t$ for $1 \leq i \leq n$, $1 \leq j \leq r$. By taking in S the homogeneous ideals $\mathcal{M}_1 = (X_1, \ldots, X_n)S$ and $\mathcal{M}_2 = (Y_1, \ldots, Y_r)S$, the Mayer-Vietoris long exact sequence gives then

$$\cdots \to H^q_{\mathcal{M}_1}(R) \oplus H^q_{\mathcal{M}_2}(R) \to H^q_{S_+}(R) \to H^{q+1}_{\mathcal{M}}(R) \to \cdots$$

Notice that for $j > c$ we have $H^q_{\mathcal{M}}(R)_{(i,j)} = 0$ for all i, q. Then, by Proposition 2.1.18 we also get $H^q_{\mathcal{M}_2}(R)_{(i,j)} = 0$ for $j > c$, for all i, q. Furthermore, $H^q_{\mathcal{M}_1}(R)_{(i,j)} = H^q_{\mathfrak{m}}(I^j)_i$ for any $j \geq 0$ by Proposition 2.1.18. Therefore, for any $j > c$ there exists an integer $i_0 = a_*(I^j)$ (depending on j) such that $H^q_{R_+}(R)_{(i,j)} = H^q_{S_+}(R)_{(i,j)} = H^q_{\mathfrak{m}}(I^j)_i = 0$ for all q and $i > i_0$. Hence $P(i,j) = \dim_k R_{(i,j)} = \dim_k (I^j)_i$ for any $j > c$ and $i > i_0$.

Now, by defining $P_j(s) = \binom{n+s-1}{n-1} - P(s,j)$, for $j > c$, $s \gg 0$ we have that $P_j(s) = \dim_k (A/I^j)_s$. Hence $P_j(s)$ is the Hilbert polynomial of A/I^j. Furthermore, we can write

$$\begin{aligned} P_j(s) &= \tbinom{n+s-1}{n-1} - P(s,j) \\ &= \tbinom{n+s-1}{n-1} - \textstyle\sum_{l+m \leq n-1} a_{lm} \tbinom{s-dj}{l} \tbinom{j}{m} \\ &= \textstyle\sum_{k=0}^{n-1} b_k(j) \tbinom{s+k}{k}, \end{aligned}$$

with $b_k(j)$ polynomials in j. Since $\deg P_j(s) = n - h - 1$ for any $j > c$, we have $b_k(j) = 0$ for $k \geq n - h$ and $j > c$, so $b_k(j) \equiv 0$ for $k \geq n - h$. Then we may write

$$P_j(s) = \sum_{k=0}^{n-h-1} (-1)^{n-h-1-k} e_{n-h-1-k}(j) \binom{s+k}{k},$$

for $j > c$. Moreover, since $P_R(s,t)$ has total degree $n - 1$ and $P_R(s,t) = \binom{n+s-1}{n-1} - P_t(s)$, we easily obtain that $\deg e_{n-h-1-k}(j) \leq n - k - 1$. $\square$

Remark 6.1.2 We have seen that $\deg e_{n-h-1-k}(j) \leq n - k - 1$ for all k, so in particular the polynomial $e_0(j)$ which gives the multiplicity of A/I^j has degree $\leq h$. By Nagata's formula,

$$e_0(j) = e(A/I^j) = \sum_{\mathfrak{p} \in \mathrm{Assh}(A/I)} \mathrm{length}(\, A_{\mathfrak{p}}/I^j_{\mathfrak{p}}) e(A/\mathfrak{p}),$$

with $\mathrm{Assh}(A/I) = \{\, \mathfrak{p} \in \mathrm{Ass}\,(A/I) \mid \dim A/\mathfrak{p} = \dim A/I\}$. Note that for all those $\mathfrak{p}$, we have that $\dim A_{\mathfrak{p}} = h$ and then

$$\mathrm{length}(\, A_{\mathfrak{p}}/I^j_{\mathfrak{p}}) = e(IA_{\mathfrak{p}}, A_{\mathfrak{p}}) \binom{h+j}{j} + \text{polynomial in } j \text{ of degree lower than } h.$$

Therefore $e_0(j)$ has degree h, so let us write

$$e_0(j) = \lambda_h \binom{j}{h} + \text{polynomial in } j \text{ of degree lower than } h.$$

We can give an upper bound for the leading coefficient λ_h. According to [HS, Corollary 3.8], we have

$$e_0(j) \leq \binom{\operatorname{reg}(I^j) + h - 1}{h}.$$

Assume that I is generated by forms in degree $\leq d$. Then there exists a positive integer α such that $\operatorname{reg}(I^j) \leq dj + \alpha$ by Theorem 3.4.6, and so $\lambda_h \leq d^h$. In Proposition 6.1.4 we will show that λ_h and, more generally, the leading coefficients of the polynomials $e_{n-h-1-k}(j)$ play an important role in the mixed multiplicities of the Rees algebra and the form ring.

Now let us consider a homogeneous ideal I generated by forms in degree d. Let us take the Hilbert polynomial $P_R(s,t)$ of its Rees algebra with the usual bigrading, and let us write

$$P_R(s+dt,t) = \sum_{k+m \leq n-1} a_{km} \binom{s}{k} \binom{t}{m}.$$

Following [HHRT], we call $e_i(R) = a_{i,n-1-i}$ the mixed multiplicity of R of type i for $i = 0, \ldots, n-1$. According to Proposition 1.5.1 we have $e_i(R) \geq 0$, and then $e(R) = \sum_{i=0}^{n-1} e_i(R)$ by [HHRT, Theorem 4.3]. Next we are going to study the multiplicity of the Rees ring and to relate it to the multiplicity of the form ring. First, we need to compute the relevant dimension of the form ring.

Lemma 6.1.3 *Let I be a homogeneous ideal in A generated by forms in degree d. Then the relevant dimension of G is n if and only if I is not $\mathfrak{m}$-primary.*

Proof. If I is $\mathfrak{m}$-primary, then $G_+ \subset P$ for any homogeneous prime $P \in \operatorname{Spec}(G)$ because $G_{(1,0)}$ is nilpotent, and so $\operatorname{rel.dim} G = 1 < \dim G = n$. If I is not $\mathfrak{m}$-primary, for $k = 0, \ldots, n-h-1$ let us write

$$e_{n-h-1-k}(j) = \lambda_{n-k-1} \binom{j}{n-k-1} + \text{polynomial in } j \text{ of lower degree.}$$

Then for $s \gg 0$, $t \gg 0$, we have

$$
\begin{aligned}
P_G(s+dt,t) &= P_{A/I^{t+1}}(s+dt) - P_{A/I^t}(s+dt)\\
&= \textstyle\sum_{k=0}^{n-h-1}(-1)^{n-h-1-k}(e_{n-h-1-k}(t+1)-e_{n-h-1-k}(t))\binom{s+dt+k}{k}\\
&= \textstyle\sum_{k=0}^{n-h-1}(-1)^{n-h-1-k}\lambda_{n-k-1}(\binom{t+1}{n-k-1}-\binom{t}{n-k-1})\binom{s+dt+k}{k} \; + \\
&\quad + \text{ polynomial in } s,t \text{ of lower total degree}\\
&= \textstyle\sum_{k=0}^{n-h-1}(-1)^{n-h-1-k}\lambda_{n-k-1}\binom{t}{n-k-2}\frac{(s+dt)^k}{k!} \; + \\
&\quad + \text{ polynomial in } s,t \text{ of lower total degree}\\
&= \textstyle\sum_{i=0}^{n-h-1}\left[\sum_{k=i}^{n-h-1}(-1)^{n-h-1-k}\lambda_{n-k-1}d^{k-i}\binom{n-2-i}{k-i}\right]\binom{s}{i}\binom{t}{n-2-i} \; + \\
&\quad + \text{ polynomial in } s,t \text{ of lower total degree}
\end{aligned}
$$

In particular, λ_h is the coefficient of $\binom{s}{n-h-1}\binom{t}{h-1}$ which is not zero by Remark 6.1.2. So the total degree of the Hilbert polynomial of the form ring is $n-2$, and then the relevant dimension of G is n by Proposition 1.5.1. □

If I is a homogeneous ideal generated by forms in degree d which is not $\mathfrak{m}$-primary, let us consider the Hilbert polynomial of its form ring

$$P_G(s+dt,t) = \sum_{k+m\leq n-2} b_{km}\binom{s}{k}\binom{t}{m}.$$

We call $e_i(G) = b_{i,n-2-i} \geq 0$ the mixed multiplicity of G of type i for $i = 0,\dots,n-2$. Then $e(G) = \sum_{i=0}^{n-2} e_i(G)$ again by [HHRT, Theorem 4.3].

Now we can give the mixed multiplicities of the Rees algebra and the form ring of an equigenerated ideal by means of the leading coefficients of the polynomials $e_{n-h-1-k}(j)$ given by Theorem 6.1.1, and to relate the mixed multiplicities of both rings.

Proposition 6.1.4 *Let I be a homogeneous ideal generated in degree d which is not $\mathfrak{m}$-primary. Set $h = \operatorname{ht}(I)$, $l = l(I)$. For each k, let us write*

$$e_{n-h-1-k}(j) = \lambda_{n-k-1}\binom{j}{n-k-1} + \text{polynomial in } j \text{ of lower degree.}$$

Then

(i) $e_i(G) = 0$ if $i \geq n-h$ or $i \leq n-l-2$. For each $n-l-2 < i < n-h$, we have

$$e_i(G) = \sum_{k=i}^{n-h-1}(-1)^{n-h-1-k}\lambda_{n-k-1}d^{k-i}\binom{n-2-i}{k-i}.$$

(ii) $e_i(R) = 0$ *if* $i \leq n-l-1$. *For each* $i > n-l-1$, *we have*

$$e_i(R) = \begin{cases} d^{n-1-i} & \text{if } i \geq n-h \\ d^{n-1-i} - \sum_{k=i}^{n-h-1}(-1)^{n-h-1-k}\lambda_{n-k-1}d^{k-i}\binom{n-1-i}{k-i} & \text{otherwise} \end{cases}$$

(iii) $e_i(G) = de_{i+1}(R) - e_i(R)$, *for* $i = 0, \ldots, n-2$. *In particular, we have* $e_i(R) \leq de_{i+1}(R)$, *for* $i = 0, \ldots, n-2$. *Furthermore,*

$$e(G) = \begin{cases} (d-1)e(R) + 1 & \textit{if } l \leq n-1 \\ (d-1)e(R) + 1 - de_0(R) & \textit{if } l = n \end{cases}$$

Proof. Let us fix $j > a_*^2(G)$. Then we have that for $s \gg 0$,

$$\dim_k \left(\frac{I^j}{I^{j+1}}\right)_{s+dj} = P_G(s+dj, j) = \sum_{k+m\leq n-2} b_{km}\binom{s}{k}\binom{j}{m},$$

so $P_G(s+dj, j)$ is the Hilbert polynomial of the A/I-module I^j/I^{j+1} for large j. Hence $b_{km} = 0$ for any $k \geq n-h$, so in particular $e_i(G) = 0$ for $i \geq n-h$.

Let us fix now $i > a_*^1(G^\varphi)$. Then we have that $H^q_{\mathcal{M}}(G)_{(i+dj,j)} = H^q_{\mathcal{M}_1}(G)_{(i+dj,j)} = 0$ for all q and j by Remark 5.2.2. From the Mayer-Vietoris long exact sequence, we have that for $t \gg 0$,

$$\dim_k \left(\frac{I^t}{I^{t+1}}\right)_{i+dt} = P_G(i+dt, t) = \sum_{k+m\leq n-2} b_{km}\binom{i}{k}\binom{t}{m}.$$

Therefore, we have that $P_G(i+dt, t)$ is the Hilbert polynomial of the $F_\mathfrak{m}(I) = k[I_d]$-module $E_i = \bigoplus_{j\geq 0}(I^j/I^{j+1})_{i+dj}$. Hence $b_{km} = 0$ for $m \geq l$. Then the first part of (i) is proved, and for the rest it suffices to notice that for $s, t \gg 0$, $P_G(s+dt, t) = P_{A/I^{t+1}}(s+dt) - P_{A/I^t}(s+dt)$.

To get (ii) and (iii), it is just enough to take into account that for $s, t \gg 0$ we have

$$P_R(s+dt, t) = P_A(s+dt) - P_{A/I^t}(s+dt),$$

$$P_G(s+dt, t) = P_R(s+dt, t) - P_R(s+dt, t+1). \ \square$$

Remark 6.1.5 Let $(A, \mathfrak{m}, k)$ be a local ring, and $I \neq A$ an ideal. Set $n = \dim A$, $l = l(I)$, $h = \operatorname{ht}(I)$. Let us denote by $\overline{G} = G_\mathfrak{m}(G_I(A))$ bigraded by means of

$$\overline{G}_{(i,j)} = \frac{\mathfrak{m}^i I^j + I^{j+1}}{\mathfrak{m}^{i+1}I^j + I^{j+1}}.$$

Since $\overline{G}$ is a standard bigraded k-algebra, we may consider its Hilbert polynomial

$$P_{\overline{G}}(s,t) = \sum_{k+m \leq n-2} c_{km} \binom{s}{k} \binom{t}{m},$$

and let us denote by $c_i(\overline{G}) = c_{i-1,n-1-i}$ for $1 \leq i \leq n-1$. R. Achilles and M. Manaresi [AM] show that $c_i(\overline{G}) = 0$ if $i > \dim A/I$ or $i < n-l$.

This definition and the results proved in [AM] can be extended to the graded case, and then we get that $c_i(\overline{G}) = 0$ if $i > n-h$ or $i < n-l$. For a homogenous ideal I in $A = k[X_1, \ldots, X_n]$ generated by forms of degree d, note that

$$\overline{G}_{(i,j)} = \left(\frac{I^j}{I^{j+1}}\right)_{i+dj} = G_{(i+dj,j)},$$

so $e_i(G) = c_{i+1}(\overline{G})$, and we get part of (i) of the previous proposition. In fact, the idea for proving this part of (i) is similar to [AM].

Remark 6.1.6 If I is a complete intersection ideal then $l = h$, and from Lemma 6.1.4 (ii) we have $e_i(R) = 0$ if $i \leq n-l-1$, and $e_i(R) = d^{n-1-i}$ if $i \geq n-l$. This result was proved in [STV, Theorem 3.6].

Corollary 6.1.7 *For a homogeneous ideal I generated by forms in degree d of height h, we have*

(i) $e(R) \geq 1 + d + \ldots + d^{h-1}$.

(ii) If I is equimultiple, $e(R) = 1 + d + \ldots + d^{h-1}$. Assume further that I is not a $\mathfrak{m}$-primary ideal. Then $e(G) = e_{n-h-1}(G) = \lambda_h = d^h$.

(iii) Assume that I is not a $\mathfrak{m}$-primary ideal and A/I^j is Cohen-Macaulay for $j \gg 0$ (or Buchsbaum). Then $e_{n-h-1}(G) = d^h$, so $e(G) \geq d^h$.

Proof. (i) and (ii) are trivial. To get (iii), according to [HRTZ, Proposition 2.3] for $j \gg 0$ we have that $e(A/I^j) \geq \binom{dj+h-2}{h}$, and so $\lambda_h \geq d^h$. Furthermore, $\lambda_h \leq d^h$ because $e_{n-h-1}(R) = d^h - \lambda_h \geq 0$ by Lemma 6.1.4. We conclude $e_{n-h-1}(G) = \lambda_h = d^h$, and so $e(G) \geq e_{n-h-1}(G) = d^h$. $\square$

Notice that as a consequence of Theorem 6.1.1 we have that with the Hilbert polynomials of a finite set of the powers of an ideal we can compute the Hilbert polynomials of its Rees algebra and its form ring, without needing an explicit presentation of these bigraded algebras. For equigenerated ideals, we may also compute the multiplicities of the Rees algebra and the form ring. Namely,

Corollary 6.1.8 *Let I be a homogeneous ideal in A. Set $c = a_*^2(R_A(I))$, $h = \operatorname{ht}(I)$. Then the Hilbert polynomials of I^j for $c+1 \leq j \leq c+n$ determine*

(i) The polynomials $e_{n-h-1-k}(j)$ for $k = 0, \dots, n-h-1$.

(ii) The Hilbert polynomials of A/I^j for $j > c+n$.

(iii) The Hilbert polynomial of the Rees algebra of I and the Hilbert polynomial of the form ring of I.

(iv) If I is equigenerated and not $\mathfrak{m}$-primary, the multiplicity of the Rees algebra of I and the multiplicity of the form ring of I.

We describe all these computations by means of an explicit example.

Example 6.1.9 Let us consider $I \subset A = k[X_1, X_2, X_3, X_4]$ the defining ideal of the twisted cubic in $\mathbb{P}^3_k$. Recall from Example 5.2.3 that the Rees algebra of I is Cohen-Macaulay, and so $a_*^2(R_A(I)) = -1$. Moreover, I is an ideal of height 2 generated by forms in degree 2. Then, according to Corollary 6.1.8 we can get the Hilbert polynomials of A/I^j for $j > 3$ from the Hilbert polynomials of I, I^2 and I^3. By using CoCoa, we have

$$\begin{aligned} P_{A/I}(s) &= 3s + 1 \\ P_{A/I^2}(s) &= 9s - 7 \\ P_{A/I^3}(s) &= 18s - 34 \end{aligned}$$

By imposing $e_0(0) = 0$, $e_0(1) = 3$ and $e_0(2) = 9$, we get the multiplicity function $e_0(t) = \frac{3}{2}t(t+1)$. Similarly, one gets $e_1(t) = \frac{5}{3}t(t+1)(t-\frac{2}{5})$. Then the Hilbert polynomial of A/I^j is

$$P_{A/I^j}(s) = e_0(j)\binom{s+1}{1} - e_1(j)$$

and the Hilbert polynomial of the Rees algebra R of I is

$$P_R(s,t) = \binom{s+3}{3} - e_0(t)\binom{s+1}{1} + e_1(t).$$

In this case $\lambda_2 = 3$, $\lambda_3 = 10$, so by Proposition 6.1.4 we have $e_3(R) = 1$, $e_2(R) = 2$, $e_1(R) = 1$, $e_0(R) = 0$, and $e_2(G) = 0$, $e_1(G) = 3$, $e_0(G) = 2$. Therefore, the multiplicity of the Rees algebra is $e(R) = \sum_{i=0}^{3} e_i(R) = 4$ and the multiplicity of the form ring is $e(G) = \sum_{i=0}^{2} e_i(G) = 5$.

6.2 Hilbert series of the powers of an ideal

Let $A = k[X_1, \ldots, X_n]$ be the polynomial ring over a field k in n variables. For any finitely generated graded A-module M, recall that the Hilbert series of M is defined as

$$H_M(s) = \sum_{j\in\mathbb{Z}} H(M,j)s^j = \sum_{j\in\mathbb{Z}} \dim_k M_j s^j \in \mathbb{Z}[[s]].$$

Following A. Conca and G. Valla [CV], for a given class $\mathcal{C}$ of homogeneous ideals in A, we say that $\mathcal{C}$ has rigid powers if for any ideals I, J in $\mathcal{C}$ such that $H_{A/I}(s) = H_{A/J}(s)$ then $H_{A/I^j}(s) = H_{A/J^j}(s)$ for all j. For example, the class of complete intersection ideals has rigid powers. The class of the homogeneous ideals in A which are Cohen-Macaulay of codimension 2 and the class of the homogeneous ideals in A which are Gorenstein of codimension 3 do not have rigid powers, but their subclasses consisting of the ideals of linear type have this property as it has been proved in [CV].

Our first aim in this section is to show that for an equigenerated ideal I we can compute the Hilbert series of A/I^j for $j \geq 1$ from a finite set among these Hilbert series, and so we can also compute the bigraded Hilbert series of its Rees algebra. This fact will be a direct consequence of the noetherian property of the Rees algebra, and the finite set of Hilbert series will be found thanks to the bounds for the shifts of the bigraded minimal free resolution of the Rees algebra given by Theorem 1.3.4. In particular, we will have that if the Hilbert series of the powers of two ideals I, J coincide for certain exponents then all the Hilbert series of the powers of I and J must coincide.

Theorem 6.2.1 *Let I be an equigenerated homogeneous ideal. Set $l = l(I)$, $c = a_*^2(R_A(I))$. The Hilbert series of I^j for $c+1 \leq j \leq c+l$ determine the Hilbert series of I^j for $j > c+l$.*

Proof. Let us assume that I is generated by forms in degree d. Then we have that $R = R_A(I)$ is a finitely generated bigraded S-module in a natural way, for $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_l]$ the polynomial ring with $\deg(X_i) = (1,0)$, $\deg(Y_j) = (d,1)$. Let $H_R(s,t)$ be the bigraded Hilbert series of R, that is

$$H_R(s,t) = \sum_{i,j} \dim_k R_{(i,j)} s^i t^j = \sum_{i,j} \dim_k (I^j)_i \; s^i t^j = \sum_j H_{I^j}(s) t^j.$$

By considering the bigraded minimal free resolution of R as S-module

$$0 \to D_t \to \ldots \to D_0 \to R \to 0,$$

$$D_p = \bigoplus_{(a,b)\in\Omega_p} S(a,b),$$

we can write

$$H_R(s,t) = \frac{Q(s,t)}{(1-s)^n(1-s^d t)^l} \ ,$$

with $Q(s,t) = \sum_{p=0}^{t}(-1)^p \sum_{(a,b)\in\Omega_p} s^{-a}t^{-b} \in \mathbb{Z}[s,t]$. Now let us fix $\alpha \in \mathbb{Z}_{\geq 0}$. Denoting by $\beta_p^j = \dim_k \operatorname{Tor}_p^A(k, I^j)_{\alpha+dj}$, then

$$\begin{aligned}
\textstyle\sum_p(-1)^p\beta_p^j &= [(1-s)^n H_{I^j}(s)\ ]_{\deg s=\alpha+dj} \\
&= [(1-s)^n H_R(s,t)\ ]_{\substack{\deg s\,=\,\alpha+dj \\ \deg t\,=\,j}} \\
&= \left[\ \tfrac{Q(s,t)}{(1-s^d t)^l}\ \right]_{\substack{\deg s\,=\,\alpha+dj \\ \deg t\,=\,j}}
\end{aligned}$$

Let us write $Q(s,t) = \sum_k m_k s^{\alpha+dk}t^k + \overline{Q}(s,t)$, with $\overline{Q}(s,t)$ containing all the monomials of the type $s^{\beta+dk}t^k$ for any $\beta \neq \alpha$ and any k. The pairs $(-\alpha-dk,-k)$ are shifts in the bigraded minimal free resolution of R as S-module, so $k \leq t_*^2(R) = a_*^2(R) + l = k_0$ by Theorem 1.3.4. Then we have that for any $j \geq k_0$,

$$\begin{aligned}
\textstyle\sum_p(-1)^p\beta_p^j &= \left[\left(\ \textstyle\sum_{v=0}^{j}\binom{v+l-1}{l-1}s^{dv}t^v\right)\left(\textstyle\sum_{k=0}^{k_0} m_k s^{\alpha+dk}t^k\right)\ \right]_{\substack{\deg s\,=\,\alpha+dj \\ \deg t\,=\,j}} \\
&= m_0\tbinom{j+l-1}{l-1} + \ldots + m_{k_0}\tbinom{j-k_0+l-1}{l-1} \\
&= P_\alpha(j).
\end{aligned}$$

It is easy to prove that this equality holds for $j \geq k_0 - l + 1 = a_*^2(R) + 1$. So we have found a polynomial $P_\alpha(j)$ of degree $\leq l-1$ such that $P_\alpha(j) = \sum_p(-1)^p\beta_p^j$ for any $j \geq a_*^2(R)+1$. Hence the Hilbert series of the powers I^j for $a_*^2(R)+1 \leq j \leq a_*^2(R)+l$ will determine the Hilbert series of I^j for any $j > a_*^2(R)+l$. $\square$

Corollary 6.2.2 *Let I be an equigenerated homogeneous ideal whose Rees algebra is Cohen-Macaulay, and $l = l(I)$. Then the Hilbert series of I^j for $j \leq l-1$ determine the bigraded Hilbert series of the Rees algebra of I.*

Recent papers by A. M. Bigatti, A. Capani, G. Niesi and L. Robbiano [BCNR] and L. Robbiano and G. Valla [RV] treat the problem of computing the Hilbert series of the powers of a homogeneous ideal I in the polynomial ring $A = k[X_1, \ldots, X_n]$. The strategy followed there to solve this problem is to compute the Rees algebra $R_A(I)$ of I and then a Gröbner basis of it, from which one can get easily the bigraded Hilbert series of the Rees algebra, and so the Hilbert series of all the powers of I. Notice that we can use Theorem 6.2.1 to give another approach to this problem: To get the Hilbert series of the powers of an equigenerated ideal I it suffices to compute the Hilbert series of $l(I)$ of its powers. Next we apply this procedure to the following example studied by A. Bigatti et al. [BCNR, Example 5.4].

Example 6.2.3 Let us consider the ideal I generated by the 2 by 2 minors of the generic symmetric 3 by 3 matrix

$$M = \begin{pmatrix} X_1 & X_2 & X_3 \\ X_2 & X_4 & X_5 \\ X_3 & X_5 & X_6 \end{pmatrix}.$$

The Rees algebra of I is Cohen-Macaulay, so $a^2_*(R) = -1$. Therefore, the Hilbert series of I^j for $j \leq 5$ will determine the rest of the Hilbert series. By using CoCoa, we obtain:

$$H_I(s) = \frac{6s^2-8s^3+3s^4}{(1-s)^6},$$

$$H_{I^2}(s) = \frac{21s^4-45s^5+38s^6-18s^7+6s^8-s^9}{(1-s)^6},$$

$$H_{I^3}(s) = \frac{56s^6-150s^7+165s^8-100s^9+36s^{10}-6s^{11}}{(1-s)^6},$$

$$H_{I^4}(s) = \frac{126s^8-385s^9+486s^{10}-330s^{11}+125s^{12}-21s^{13}}{(1-s)^6},$$

$$H_{I^5}(s) = \frac{252s^{10}-840s^{11}+1155s^{12}-840s^{13}+330s^{14}+56s^{15}}{(1-s)^6}.$$

Then the polynomials $P_\alpha(j)$ defined in the proof of Theorem 6.2.1 are

$$P_\alpha(j) = 0 \text{ for } \alpha \neq 0, \ldots, 5 \ ,$$

$$P_0(j) = 1 + \tfrac{137}{60} j + \tfrac{15}{8} j^2 + \tfrac{17}{24} j^3 + \tfrac{1}{8} j^4 + \tfrac{1}{20} j^5,$$

$$P_1(j) = -\tfrac{7}{4}j - \tfrac{83}{24}j^2 - \tfrac{53}{24}j^3 - \tfrac{13}{24}j^4 - \tfrac{1}{24}j^5,$$

$$P_2(j) = -\tfrac{3}{2}j + \tfrac{13}{12}j^2 + \tfrac{29}{12}j^3 + \tfrac{11}{12}j^4 + \tfrac{1}{12}j^5,$$

$$P_3(j) = \tfrac{7}{6}j + \tfrac{3}{4}j^2 - \tfrac{13}{12}j^3 - \tfrac{3}{4}j^4 - \tfrac{1}{12}j^5,$$

$$P_4(j) = -\tfrac{1}{4}j - \tfrac{7}{24}j^2 + \tfrac{5}{24}j^3 + \tfrac{7}{24}j^4 + \tfrac{1}{24}j^5,$$

$$P_5(j) = \tfrac{1}{20}j + \tfrac{1}{24}j^2 - \tfrac{1}{24}j^3 - \tfrac{1}{24}j^4 - \tfrac{1}{120}j^5,$$

and the Hilbert series of I^j is

$$\frac{P_0(j)s^{2j} + P_1(j)s^{2j+1} + P_2(j)s^{2j+2} + P_3(j)s^{2j+3} + P_4(j)s^{2j+4} + P_5(j)s^{2j+5}}{(1-s)^6}.$$

Next we also compute the Hilbert series of the powers of the ideal of the twisted cubic in $\mathbb{P}^3_k$ studied in the previous section.

Example 6.2.4 Let $I \subset A = k[X_1, X_2, X_3, X_4]$ be the defining ideal of the twisted cubic in $\mathbb{P}^3_k$. From Example 5.2.3, let us recall that I is generated by quadrics with $l(I) = \mu(I) = 3$ and $R_A(I)$ is Cohen-Macaulay. Therefore, according to Theorem 6.2.1, we can get the Hilbert series of I^j for $j > 2$ from the Hilbert series of I and I^2. By using CoCoa, we have

$$H_I(s) = \tfrac{3s^2-2s^3}{(1-s)^4}\ ,$$

$$H_{I^2}(s) = \tfrac{6s^4-6s^5+s^6}{(1-s)^4}.$$

Then the polynomials $P_\alpha(j)$ defined in the proof of Theorem 6.2.1 are

$$P_\alpha(j) = 0, \text{ for } \alpha \neq 0, 1, 2\ ,$$

$$P_0(j) = \tfrac{1}{2}(j+1)(j+2)\ ,$$

$$P_1(j) = -j(j+1)\ ,$$

$$P_2(j) = \tfrac{1}{2}j(j-1)\ ,$$

and the Hilbert series of I^j is then

$$H_{I^j}(s) = \frac{P_0(j)s^{2j} + P_1(j)s^{2j+1} + P_2(j)s^{2j+2}}{(1-s)^4}.$$

Now, the bigraded Hilbert series of its Rees algebra is

$$H_R(s,t) = \sum_j H_{I^j}(s)t^j = \frac{1 - 2s^3t + s^6t^2}{(1-s)^4(1-s^2t)^3}.$$

Remark 6.2.5 Similarly we can prove the following statement for the Hilbert series of the form ring of an equigenerated ideal I: If $l = l(I)$ and $e = a_*^2(G_A(I))$, then the Hilbert series of I^j/I^{j+1} for $e+1 \le j \le e+l$ determine the Hilbert series of I^j/I^{j+1} for $j > e+l$. In fact, for any $a \ge e$ the Hilbert series of I^j/I^{j+1} for $a+1 \le j \le a+l$ determine the rest.

For any $m \ge 0$, let us define C_m to be the class of equigenerated homogeneous ideals I in A such that $a_*^2(G_A(I)) + l(I) \le m$. Note that C_0 contains the class of complete intersection ideals, and we have the chain

$$C_0 \subset C_1 \subset \cdots \subset C_m \subset C_{m+1} \subset \cdots$$

As a corollary, we get

Corollary 6.2.6 *Let $I, J \in C_m$ be such that*

$$H_{I^j/I^{j+1}}(s) = H_{J^j/J^{j+1}}(s), \text{ for } m-l+1 \le j \le m.$$

Then $H_{I^j/I^{j+1}}(s) = H_{J^j/J^{j+1}}(s)$, $\forall j$. Therefore $H_{I^j}(s) = H_{J^j}(s)$, for all j, and in particular C_0 has rigid powers.

For an arbitrary homogeneous ideal I in A, we can also show that a finite set of Hilbert series of the powers of I determine the rest. But in this case, the bound we get is worse.

Proposition 6.2.7 *Let I be a homogeneous ideal in A. Set $r = \mu(I)$, $c = a_*^2(R_A(I))$. The Hilbert series of I^j for $j \le c+r$ determine the Hilbert series of I^j for $j > c+r$.*

Proof. Assume that I is minimally generated by forms $f_1,...,f_r$ of degrees $d_1, \ldots, d_r$ respectively. Then let us consider the presentation of the Rees algebra R of I as a quotient of the polynomial ring $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$, with $\deg(X_i) = (1,0)$, $\deg(Y_j) = (d_j, 1)$. From the bigraded minimal free resolution of R as S-module, we have that there is a polynomial $Q(s,t) \in \mathbb{Z}[s,t]$ such that

$$H_R(s,t) = \frac{Q(s,t)}{(1-s)^n(1-s^{d_1}t)\ldots(1-s^{d_r}t)} \quad .$$

According to Theorem 1.3.4 we can write $Q(s,t) = \sum_{i=0}^{m} Q_i(s)t^i$, with $m = a_*^2(R) + r$. Since $H_R(s,t) = \sum_{j\geq 0} H_{I^j}(s)t^j$, we have

$$Q(s,t) = (1-s)^n(1-s^{d_1}t)\dots(1-s^{d_r}t)(\sum_{j\geq 0} H_{I^j}(s)t^j),$$

and then the result follows immediately. □

6.3 Minimal graded free resolutions of the powers of an ideal

The results about the Hilbert series of the powers of a homogeneous ideal imply in some particular cases the estability (in a meaning which we will precise immediately) of the minimal graded free resolutions of the powers of the ideal. For instance,

Proposition 6.3.1 *Let I be an ideal generated in degree d with $l(I) = 2$ whose Rees algebra is Cohen-Macaulay. Then the minimal graded free resolution of I determines the minimal graded free resolutions of all its powers. Namely, if the minimal graded free resolution of I is*

$$0 \to A(-\alpha_1 - d)^{\beta_1} \oplus \dots \oplus A(-\alpha_m - d)^{\beta_m} \to A(-d)^{\beta} \to I \to 0,$$

then for any $j \geq 1$ the minimal graded free resolution of I^j is

$$0 \to A(-\alpha_1 - dj)^{\beta_1 j} \oplus \dots \oplus A(-\alpha_m - dj)^{\beta_m j} \to A(-dj)^{(\beta-1)j+1} \to I^j \to 0.$$

Proof. First, note that for any $j \geq 1$ we have that $\text{proj.dim}_A I^j \leq l(I) - 2 = 1$ because R is Cohen-Macaulay (see Proposition 6.3.2). Therefore, we can conclude that $\text{proj.dim}_A I^j = 1$ for any j. On the other hand, since R is Cohen-Macaulay, we have $a_*^2(R) = -1$, so the Hilbert series of I^j for $j \leq l - 1 = 1$ determine the Hilbert series of I^j for $j > 1$ according to Theorem 6.2.1. The polynomials $P_\alpha(j)$ defined there are

$$P_\alpha(j) = 0, \text{ for } \alpha \notin \{0, \alpha_1, \dots, \alpha_m\},$$

$$P_0(j) = (\beta - 1)j + 1,$$

$$P_{\alpha_i}(j) = -\beta_i\, j, \text{ for } i \in \{0, \dots, m\}.$$

Then the Hilbert series of I^j is

$$H_{I^j}(s) = \frac{\sum_\alpha P_\alpha(j) s^{\alpha+dj}}{(1-s)^n},$$

so the minimal graded free resolution of I^j must be

$$0 \to A(-\alpha_1 - dj)^{-P_{\alpha_1}(j)} \oplus \ldots \oplus A(-\alpha_m - dj)^{-P_{\alpha_m}(j)} \to A(-dj)^{P_0(j)} \to I^j \to 0.$$

□

This result leads to the question of when a finite number of minimal graded free resolutions of the powers of I determine the rest (and, in this case, which set of resolutions determine the others).

Let us begin by studying the behaviour of the projective dimension of the powers of I. It is well-known that these projective dimensions are asymptotically constant (see [Bro, Theorem 2]), but not for which powers of the ideal the projective dimension takes the asymptotic value. We will precise these powers for ideals whose form ring is Gorenstein by considering the Koszul homology of the Rees algebra R of I with respect to $X_1, \ldots, X_n$. This also provides new proofs of well-known results as the Burch's inequality or the constant asymptotic value for the depth.

Proposition 6.3.2 *Let I be a homogeneous ideal in A, and set $l = l(I)$. Then:*

(i) $\mathrm{proj.dim}_A(I^j) \leq n - \mathrm{depth}_{(\mathfrak{m}R)}(R)$ *for all j, and the equality holds for $j \gg 0$. So,* $\inf_{j\geq 0}\{\mathrm{depth}(A/I^j)\} = n - l - (\mathrm{ht}\,(\mathfrak{m}R) - \mathrm{depth}_{(\mathfrak{m}R)}(R))$.

(ii) If R is Cohen-Macaulay, $\mathrm{proj.dim}_A(I^j) \leq l - 1$ *for any j and* $\mathrm{proj.dim}_A(I^j) = l - 1$ *for $j \gg 0$. Furthermore,* $\mathrm{proj.dim}_A(I^j) = l - 1$ *implies* $\mathrm{proj.dim}_A(I^{j+1}) = l - 1$.

(iii) If G is Gorenstein, $\mathrm{proj.dim}_A I^j = l - 1$ *if and only if $j > a^2(G) - a(F)$, and* $\mathrm{proj.dim}_A I^j/I^{j+1} = l$ *if and only if $j \geq a^2(G) - a(F)$.*

Proof. Let us consider the Koszul complex $K.(\underline{X}, R) = K.(X_1, \ldots, X_n, R)$ of the Rees algebra R with respect to $\underline{X} = X_1, \ldots, X_n$. We have the natural bigrading in the Rees algebra R by means of $R_{(i,j)} = (I^j)_i$, and then the modules $K_p(\underline{X}, R)$ of the Koszul complex are also bigraded in a natural way. Denoting by $F = F_{\mathfrak{m}}(I)$, we have that for any p the Koszul homology module $H_p = H_p^S(\underline{X}; R)$ is a finitely generated bigraded F-module. Moreover, since $K_p(\underline{X}, R)_{(i,j)} = K_p(\underline{X}, I^j)_i$ we have

$$H_p^S(\underline{X}, R)_{(i,j)} = H_p^A(\underline{X}, I^j)_i = \mathrm{Tor}\,_p^A(k, I^j)_i\ ,$$

so the Koszul homology modules H_p contain all the information about the graded minimal free resolutions of the powers of I.

Now set $s = n - \text{depth}_{(\mathfrak{m}R)}(R)$. Recall that H_p is zero for any $p > s$, and so $\text{proj.dim}_A(I^j) \leq s$ for any j. Moreover, since H_s is a F-module of dimension l [Hu2, Remark 1.5] we can find for any $j \gg 0$ an integer i (depending on j) such that $[H_s]_{(i,j)} \neq 0$. Therefore we obtain that $\text{proj.dim}_A(I^j) = s$ for $j \gg 0$. By the graded Auslander-Buchsbaum formula, $\text{proj.dim}_A(I^j) = n - \text{depth}\, A/I^j - 1$, and noting that $\text{depth}_{(\mathfrak{m}R)}(R) \leq \text{ht}(\mathfrak{m}R) = n + 1 - l$ we get (i).

To prove (ii), let us denote by $t = \text{depth}_{(\mathfrak{m}R)}(R) = n + 1 - l$. We may assume that k is infinite, and then there exists a homogeneous regular sequence $b_1, \ldots, b_t \in \mathfrak{m}R$ of degree $(1, 0)$. Then

$$H_s = \frac{(b_1, \ldots, b_t) : (\mathrm{X}_1, \ldots, \mathrm{X}_n)}{(b_1, \ldots, b_t)}(t - n, 0).$$

Note that $s = l - 1$ because R is CM. Now, observe that $\text{proj.dim}_A(I^j) = l - 1$ implies that there exists i such that $[H_s]_{(i,j)} \neq 0$; so let us take $f \in [H_s]_{(i,j)}, f \neq 0$. For a positively graded ring $A = \bigoplus_{j \geq 0} A_j$, let us denote by $A_+ = \bigoplus_{j > 0} A_j$, and in the following let us consider the fiber cone F and the Rees algebra R graded by means of $F_j = I^j/\mathfrak{m}I^j$, $R_j = I^j$. If $F_+ f = 0$, then $I^m f \subset (b_1, \ldots, b_t)$ for any $m \geq 1$. So, denoting by $\overline{R} = R/(b_1, \ldots, b_t)$, we have that $\overline{R}_+ \subset \text{Ann}(f) \subset \mathfrak{p} \in \text{Ass}\,(\overline{R})$ and so $\text{ht}(\overline{R}_+) = 0$. But $\text{ht}(\overline{R}_+) = \dim \overline{R} - \dim \overline{R}/\overline{R}_+ = 1$. As a consequence, $(F_+)_1 f \neq 0$ and so there exists d such that $[H_s]_{(i+d,j+1)} \neq 0$ and we have (ii).

Finally, we are going to determine the powers of I whose projective dimension is $l-1$ if G is Gorenstein. To this end, let us consider the Koszul homology modules of the form ring G with respect to $\underline{X}$, which will be also denoted by H_p. Set $s = n - \text{depth}_{(\mathfrak{m}G)}(G) = l$, $t = \text{depth}_{(\mathfrak{m}G)}(G)$. As before, H_p is zero for $p > s$ and there exists a homogeneous regular sequence $b_1, \ldots, b_t \in \mathfrak{m}G$ of degree $(1, 0)$, such that

$$H_l = \frac{(b_1, \ldots, b_t) : (\mathrm{X}_1, \ldots, \mathrm{X}_n)}{(b_1, \ldots, b_t)}(t - n, 0).$$

On the other hand, from the natural bigraded epimorphism $G \to G/\mathfrak{m}G = F$, we can compute the canonical module of the fiber cone F by using Corollary 1.2.2 :

$$K_F = \underline{\text{Ext}}_G^{n-l}(F, K_G).$$

Since G is Gorenstein, we have a bigraded isomorphism $K_G \cong G(-n, a)$ with $a = a^2(G)$ by Corollary 4.1.7. Therefore,

$$\begin{aligned} K_F &= \underline{\mathrm{Ext}}_G^{n-l}(F, G)(-n, a) \\ &= \underline{\mathrm{Hom}}_G(F, G/(b_1, \ldots, b_t))(-n+t, a) \\ &= \tfrac{(b_1,\ldots,b_t):(X_1,\ldots,X_n)}{(b_1,\ldots,b_t)}(-n+t, a) \\ &= H_l(0, a). \end{aligned}$$

Now, observe that $\mathrm{proj.dim}_A(I^j/I^{j+1}) = l$ if and only if there exists i such that $[H_l]_{(i,j)} \neq 0$ if and only if there exists i such that $[K_F]_{(i,j-a)} \neq 0$, that is, $j \geq a - a(F)$. From the exact sequences

$$0 \to I^{j+1} \to I^j \to I^j/I^{j+1} \to 0 \ ,$$

it is then easy to check that $\mathrm{proj.dim}_A(I^j) = l-1$ if and only if $j > a - a(F)$, and so we are done.□

Example 6.3.3 Let I be a strongly Cohen-Macaulay ideal such that $\mu(I_{\mathfrak{p}}) \leq \mathrm{ht}\,(\mathfrak{p})$ for any prime ideal $\mathfrak{p} \supseteq I$. Set $l = l(I)$, $h = \mathrm{ht}(I)$. Recall from Corollary 5.2.11 that $G_A(I)$ is Gorenstein, $a^2(G_A(I)) = -h$ and $a(F_{\mathfrak{m}}(I)) = -l$. So, by Proposition 6.3.2 we have $\mathrm{depth}(A/I^j) = n - l$ if and only if $j > l - h$.

Example 6.3.4 Let $\mathbf{X} = (X_{ij})$ be a generic matrix, with $1 \leq i \leq d$, $1 \leq j \leq n$ and $d \leq n$. Let $I \subset A = k[\mathbf{X}]$ be the ideal generated by the maximal minors of $\mathbf{X}$. Recall from Example 5.2.10 that the Rees algebra R is Cohen-Macaulay and the form ring G is Gorenstein with $a^2(G) = -\mathrm{ht}\,(I) = -(n-d+1)$. Furthermore, $l(I) = d(n-d)+1$ and $a(F) = -n$. Now by Proposition 6.3.2, we get that $\mathrm{depth}(A/I^j) = d^2 - 1$ if and only if $j > d-1$. In the case $n = d+1$, this was proved in [BV, Example 9.27].

Example 6.3.5 Let $\mathbf{X} = (X_{ij})$ be a generic skew-symmetric matrix, with $1 \leq i < j \leq n$, and n odd. Let $I \subset A = k[\mathbf{X}]$ be the ideal generated by the $(n-1)$-pfaffians of $\mathbf{X}$, where k is a field. In this case, the form ring G is Gorenstein [CD] and $l(I) = n$, $a(F_{\mathfrak{m}}(I)) = -n$ [Hu3]. So $\mathrm{depth}(A/I^j)$ takes the asymptotic value $\frac{n(n-1)}{2} - n$ for some $j \leq n$, and by Proposition 6.3.2 for all $j \geq n$.

G. Boffi and R. Sánchez [BoSa] have constructed a family of complexes which give a resolution for all the powers I^j, for $j \geq 1$, in particular proving

that $\mathrm{proj.dim}_A(A/I^j) = n$ if and only if $j \geq n-2$. Then Proposition 6.3.2 shows that $a^2(G_A(I)) = -3$.

Our next aim is to study the graded minimal free resolutions of the powers of an equigenerated ideal by doing a deeper study of the Koszul homology of the Rees algebra with respect to $X_1, \ldots, X_n$. The general case will be studied later by different methods.

6.3.1 Case study : Equigenerated ideals

First of all, we show that the shifts in the graded minimal free resolutions of the powers of an equigenerated ideal are given by linear functions asymptotically and the graded Betti numbers of these resolutions are given by polynomials asymptotically.

Proposition 6.3.6 *Let I be an ideal generated by forms in degree d. Set $l = l(I)$, $s = n - \mathrm{depth}_{(\mathfrak{m}R)}(R)$. Then there is a finite set of integers*

$$\{\alpha_{pi} \mid 0 \leq p \leq s, 1 \leq i \leq k_p\}$$

and polynomials of degree $\leq l-1$

$$\{Q_{\alpha_{pi}}(j) \mid 0 \leq p \leq s, 1 \leq i \leq k_p\}$$

such that the graded minimal free resolution of I^j for j large enough is

$$0 \to D_s^j \to \ldots \to D_0^j \to I^j \to 0\ ,$$

with $D_p^j = \bigoplus_i A(-\alpha_{pi} - dj)^{\beta_{pi}^j}$ and $\beta_{pi}^j = Q_{\alpha_{pi}}(j)$.

Proof. Let us consider again the Koszul homology of the Rees algebra R of I with respect to $\underline{X} = X_1, \ldots, X_n$, and let us denote by $F = \bigoplus_{j\geq 0} I^j/\mathfrak{m}I^j$ the fiber cone of I and by $F_+ = \bigoplus_{j>0} I^j/\mathfrak{m}I^j$. For every $p \leq s$, $H_p = H_p^S(\underline{X}; R)$ is a finitely generated bigraded F-module. Let g be a homogeneous generator of H_p with $\deg(g) = (a, b)$, and set $\alpha = a - db$. If $F_+ \subset \mathrm{rad}\,(\mathrm{Ann}\,(g))$, there exists j such that $F_+^j g = 0$, and so $F_j g = 0$ for all $j \gg 0$. Otherwise, there exists a homogeneous element $f \in F$ of degree d such that $f \notin \mathrm{rad}\,(\mathrm{Ann}\,(g))$. Then $f^j g \neq 0$ for all j, and so we have $[H_p]_{(\alpha+dj,j)} \neq 0$ for all $j \gg 0$. Let $g_1, \ldots, g_m$ be the homogeneous generators of H_p with this property, and set

$\deg(g_i) = (a_i, b_i)$, $\alpha_i = a_i - db_i$. Then, for j large enough we have that $[H_p]_{(a,j)} \neq 0$ if, and only if, there exists $i \in \{1, \dots, m\}$ such that $a = \alpha_i + dj$. Since $[H_p]_{(\alpha_i+dj,j)} = \operatorname{Tor}_p^A(k, I^j)_{\alpha_i+dj}$, we obtain that $\alpha_i + dj$, for $1 \leq i \leq m$, are the only shifts in the place p of the graded minimal free resolution of I^j for $j \gg 0$.

For $\alpha \in \{\alpha_1, \dots, \alpha_m\}$, let us define $H_p^\alpha = \bigoplus_j [H_p]_{(\alpha+dj,j)}$. Notice that $\dim H_p^\alpha \leq \dim H_p = l$ by [Hu2, Remark 1.5]. Since H_p^α is a finitely generated graded F-module, there exists a polynomial $Q_\alpha(j)$ of degree $\dim H_p^\alpha - 1 \leq l-1$ such that for j large enough

$$Q_\alpha(j) = \dim_k [H_p^\alpha]_j = \dim_k \operatorname{Tor}_p^A(k, I^j)_{\alpha+dj},$$

so $Q_\alpha(j)$ is the Betti number of I^j corresponding to $\alpha + dj$ in the place p. $\square$

Example 6.3.7 Let I be a Cohen-Macaulay homogeneous ideal of codimension two in the polynomial ring $A = k[X_1, \dots, X_n]$ such that:

(i) The entries of the Hilbert-Burch matrix of I are linear forms.

(ii) I verifies G_n.

(iii) $\mu(I) \leq n$.

This example has been studied by A. Conca and G. Valla in [CV]. Set $r = \mu(I)$, $d = r - 1$, $S = A[Y_1, \dots, Y_r]$. Then

$$R_A(I) \cong \operatorname{Sym}_A(I) \cong S/(F_1, \dots, F_{r-1}),$$

with $F_1, \dots, F_{r-1}$ a regular sequence of degree $(d, 1)$. So the Koszul complex of S with respect to $F_1, \dots, F_{r-1}$ gives the bigraded minimal free resolution of $R_A(I)$. From this resolution one can get the minimal graded free resolutions of I^j, for all $j \geq 0$. Namely, $\text{proj.dim}_A(I^j) = \min\{j, r-1\}$ and the minimal free resolution of I^j is

$$0 \to D_{r-1}^j \to \dots \to D_0^j \to I^j \to 0 \ ,$$

with $D_p^j = A(-p-dj)^{\beta_p^j}$, $\beta_p^j = \binom{r-1}{p}\binom{r+j-p-1}{r-1}$.

Remark 6.3.8 Let $(A, \mathfrak{m}, k)$ be a noetherian local ring, and let $I \subset A$ be an ideal. Set $l = l(I)$, $r = \mu(I)$. Denote by R the Rees algebra of I graded by $R_j = I^j$, and let $S = A[Y_1, \dots, Y_r]$ be a polynomial ring over A with $\deg Y_j = 1$ so that R is a finitely generated graded S-module. As above, we can prove

that there are polynomials $Q_p(j)$ of degree $\leq l-1$, the Hilbert polynomials of $\mathrm{Tor}_p^S(S/\mathfrak{m}S, R)$, such that the minimal free resolutions of I^j for $j \gg 0$ are

$$\ldots \to A^{Q_p(j)} \to \ldots \to A^{Q_0(j)} \to I^j \to 0.$$

This result was proved by V. Kodiyalam in [Ko1]. If A is regular, let $\underline{x}$ be a regular sequence generating $\mathfrak{m}$. Since $\mathrm{Tor}_p^S(S/\mathfrak{m}S, R) \cong H_p(\underline{x}, R)$, the module $\mathrm{Tor}_p^A(k, R)$ has dimension l if it is not zero. Therefore, the polynomial $Q_p(j)$ has degree $l-1$ if $Q_p(j) \neq 0$. This answers affirmatively [Ko1, Question 13] for any regular ring A.

Observe that Proposition 6.3.6 says that we can compute the graded minimal free resolution of any power of an equigenerated ideal from a finite set among these resolutions. Now we consider the problem of determining this finite set of resolutions. To begin with, let us study the asymptotic shifts of Proposition 6.3.6.

Lemma 6.3.9 *Let I be a homogeneous ideal generated in degree d and let $R = R_A(I)$. Then*

(i) For all p and i, there exists $(a,b) \in \Omega_{p,R}$ such that $\alpha_{pi} = db - a$.

(ii) For each α, let

$$p = \min\{\, q \mid \exists\, b\; s.t.\, (\alpha + db, b) \in \Omega_{q,R}\},$$

and let

$$b_0 = \max\{\, b \mid (\alpha + db, b) \in \Omega_{p,R}\}.$$

Then $\alpha + db_0 \in \Omega_{p,I^{-b_0}}$, that is, $\alpha + db_0$ is a shift that appears in the graded minimal free resolution of I^{-b_0} at the place p.

Proof. Let $0 \to D_m \to \ldots \to D_0 \to R \to 0$ be the bigraded minimal free resolution of R over $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_l]$. By applying the functor $(\ )^j$ to this resolution, we get a graded free resolution of I^j over A

$$0 \to D_m^j \to \ldots \to D_1^j \to D_0^j \to I^j \to 0\,,$$

with $D_p^j = \bigoplus_{(a,b)\in\Omega_{p,R}} A(a - db - dj)^{\rho_{ab}^j}$, for some $\rho_{ab}^j \in \mathbb{Z}$. This resolution is the direct sum of the minimal graded free resolution of I^j and the trivial complex [Eis, Exercise 20.1], so we obtain that for $j \gg 0$

$$\{\alpha_{pi} + dj\}_{p,i} \subset \{db - a + dj \mid (a,b) \in \Omega_{p,R}\},$$

and so (i) is already shown.

Now let α be such that there exists b with $(\alpha+db, b) \in \Omega_R$. Let p be the first integer such that $(\alpha+db, b) \in \Omega_{p,R}$, and let b_0 be the maximum of these b's. We must show that

$$\operatorname{Tor}_p^S(S/\mathfrak{m}S, R)_{(-\alpha-db_0,-b_0)} = \operatorname{Tor}_p^A(k, I^{-b_0})_{-\alpha-db_0} \neq 0.$$

We will proceed as in Theorem 5.1.1: Let

$$D_{p+1} \xrightarrow{\psi_{p+1}} D_p \xrightarrow{\psi_p} D_{p-1}$$

be the differential maps appearing in the resolution of R. Tensorazing by $\otimes_S S/\mathfrak{m}S$, we have the sequence

$$D_{p+1}/\mathfrak{m}D_{p+1} \xrightarrow{\overline{\psi}_{p+1}} D_p/\mathfrak{m}D_p \xrightarrow{\overline{\psi}_p} D_{p-1}/\mathfrak{m}D_{p-1}.$$

Now let $v \in D_p$ be an element of the homogeneous basis of D_p as free S-module with $\deg(v) = (-\alpha-db_0, -b_0)$. If $w_1, \ldots, w_s$ is the basis of D_{p-1}, we can write

$$\psi_p(v) = \sum_{j=1}^{s} \lambda_j w_j,$$

with $\lambda_j \in \mathcal{M}$ homogeneous. Set $\deg(w_j) = (-\alpha_j - db_j, -b_j)$. By looking at the degree of the elements, we have that λ_j must be zero for all j such that $-b_j > -b_0$. For the integers j such that $-b_j = -b_0$, we have that $\lambda_j \in \mathfrak{m}S$ necessarily. Finally, for j such that $-b_j < -b_0$ we also have $\lambda_j \in \mathfrak{m}S$ because $\alpha_j \neq \alpha$. We may conclude $\overline{\psi}_p(v) = 0$, that is, $v \in \operatorname{Ker}\overline{\psi}_p$. It is clear that $v \notin \operatorname{Im}\overline{\psi}_{p+1}$ because $\operatorname{Im}\overline{\psi}_{p+1} \subset \mathcal{M}D_p$. So $v \in \operatorname{Tor}_p^S(S/\mathfrak{m}S, L)_{(-\alpha-db_0,-b_0)}$, $v \neq 0$ and we are done. $\square$

As a consequence of this lemma we have that all the differences $a - db$ for $(a,b) \in \Omega_R$ appear in the minimal graded free resolution of some power I^j of I for $j \leq a_*^2(R) + l(I)$. The problem is to distinguish which of these shifts will appear asymptotically, and the place from where on the resolutions are stable. We can solve this problem for ideals with a very particular nice behaviour. For instance, we get

Proposition 6.3.10 *Let I be an equigenerated homogeneous ideal, and set $b = a_*^2(R_A(I)) + l(I)$. If the graded minimal free resolutions of $I, I^2, \ldots, I^b$ are linear, then the graded minimal free resolutions of I^j are also linear for any j. Furthermore, the minimal free resolutions of $I, I^2, \ldots, I^b$ determine the minimal graded free resolutions of I^j for any j.*

Proof. Assume that I is generated by forms in degree d and set $s = \sup_{j=1,\ldots,b}\{\text{proj.dim}_A I^j\}$. According to Lemma 6.3.9, we have that the shifts in Ω_R are of the type $(p+db, b)$ with $0 \leq p \leq s$. Furthermore, there exists b_0 such that $(p+db_0, b_0) \in \Omega_{p,R}$, but for any b, $q < p$, $(p+db, b) \notin \Omega_{q,R}$. Therefore, $\Omega_{p,R}$ has only shifts of the form $(a+db, b)$ for $0 \leq a \leq p$. Again by Lemma 6.3.9, we get

$$\{\alpha_{pi}\}_i \subset \{0, \ldots, p\}.$$

Finally, since $\min\{-\beta : \beta \in \Omega_{p+1,I^j}\} > \min\{-\beta : \beta \in \Omega_{p,I^j}\}$, we have that $\min\{-\beta : \beta \in \Omega_{p,I^j}\} \geq p + dj$. Therefore I^j must have a linear minimal free resolution.

Moreover, by Theorem 6.2.1 we also have that for $p = 0, \ldots, s$ there exists a polynomial $Q_p(j)$ of degree $\leq l-1$ such that

$$Q_p(j) = \dim_k \operatorname{Tor}_p^A(k, I^j)_{p+dj},$$

for $j \geq a_*^2(R) + 1$. So, if we know the minimal graded free resolutions of $I^{b-l+1}, \ldots, I^b$, we may determine the polynomials $Q_p(j)$, and then the minimal graded free resolution of I^j for $j > b$.$\square$

Remark 6.3.11 The first part of Proposition 6.3.10 can be also obtained from Theorem 5.2.1 (*ii*).

Remark 6.3.12 Given an equigenerated homogeneous ideal I with a linear minimal free resolution, it can happen that I^2 has a non linear minimal free resolution (see [Con, Remark 3]). We have shown in Proposition 6.3.10 that if certain powers of I have linear resolution, then the rest of the powers have this property too.

We may apply this result to guess the minimal graded free resolutions of the powers of the ideal defining the twisted cubic in $\mathbb{P}_k^3$.

Example 6.3.13 Let $I \subset A = k[X_1, X_2, X_3, X_4]$ be the defining ideal of the twisted cubic in $\mathbb{P}_k^3$, and let us study the graded minimal free resolutions of its powers. I is generated by forms in degree 2 with $l(I) = 3$ and $b = a_*^2(R) + l(I) = 2$. The minimal resolutions of I and I^2 (computed with CoCoa) are:

$$0 \to A(-3)^2 \to A(-2)^3 \to I \to 0\ ,$$

$$0 \to A(-6) \to A(-5)^6 \to A(-4)^6 \to I^2 \to 0\ .$$

Since these resolutions are linear, we have that the minimal graded free resolutions of I^j for $j > 2$ are also linear by Proposition 6.3.10, and we may compute them:

$$0 \to A(-2-2j)^{Q_2(j)} \to A(-1-2j)^{Q_1(j)} \to A(-2j)^{Q_0(j)} \to I^j \to 0 \ ,$$

with $Q_0(j) = \frac{1}{2}(j+1)(j+2)$, $Q_1(j) = j(j+1)$ and $Q_2(j) = \frac{1}{2}j(j-1)$.

Similarly, one can prove the following statement.

Proposition 6.3.14 *Let I be an ideal generated in degree d, and set $b = a_*^2(R_A(I)) + l(I)$. Assume that there are integers $\alpha_1, \ldots, \alpha_s$ such that the graded minimal free resolutions of $I, I^2, \ldots, I^b$ take the form*

$$0 \to D_s^j \to \ldots \to D_1^j \to D_0^j \to I^j \to 0,$$

with $D_p^j = A(-\alpha_p - dj)^{\beta_p^j}$ and $\beta_p^j \geq 0$. Then the graded minimal free resolutions of I^j are of this type too. Furthermore, the minimal graded free resolutions of $I, I^2, \ldots, I^b$ determine the minimal graded free resolutions of I^j for any j.

The following example does not belong to the family of ideals considered in the previous propositions, but we can also guess the asymptotic resolution of its powers.

Example 6.3.15 Let $I = (X^7, Y^7, X^6Y + X^2Y^5) \subset A = k[X,Y]$. Note that I is a $\mathfrak{m}$-primary ideal generated by forms of degree 7 with $l(I) = 2$. Since $\text{proj.dim}_A I^j = 1$ for any $j \geq 1$, we have that the shifts in the place 0 and 1 of the resolution of I^j can not coincide. Then, according to Theorem 6.2.1 we have that for any $\alpha \neq 0$ there is a polynomial $P_\alpha(j)$ of degree ≤ 1 such that

$$P_\alpha(j) = \dim_k \operatorname{Tor}_1^A(k, I^j)_{\alpha+dj},$$

for all $j \geq a_*^2(R) + 1$.

This example was studied by S. Huckaba and T. Marley [HM, Example 3.13]. Denoting by $G_{++} = \bigoplus_{j>0} I^j/I^{j+1}$ and by $\underline{a}_i(G) = \max\{j \mid H^i_{G_{++}}(G)_j \neq 0\}$, it was proved that the form ring G has depth 0, and $\underline{a}_0(G) < \underline{a}_1(G) < \underline{a}_2(G) = 4$. Now $a_*^2(G) = \max\{\underline{a}_i(G) : i = 0,1,2\} = 4$ according to [Hy, Lemma 2.3], and then the short exact sequences

$$0 \to R_{++} \to R \to A \to 0,$$

$$0 \to R_{++}(1) \to R \to G \to 0,$$

where $R_{++} = \bigoplus_{j>0} I^j$, show $a_*^2(R) = 4$. The graded minimal free resolutions of I^5 and I^6 (computed with CoCoa) are :

$$0 \to A(-37)^{15} \oplus A(-36)^5 \to A(-35)^{21} \to I^5 \to 0\ ,$$

$$0 \to A(-44)^{15} \oplus A(-43)^{12} \to A(-42)^{28} \to I^6 \to 0\ ,$$

Then we may compute the polynomials $P_\alpha(j)$, so the graded minimal free resolutions of I^j for $j \geq 5$ are:

$$0 \to A(-2-7j)^{15} \oplus A(-1-7j)^{7j-30} \to A(-7j)^{7j-14} \to I^j \to 0\ .$$

Furthermore, in this case we check that the bound can not be improved because the resolution of I^4 is

$$0 \to A(-30)^{14} \to A(-28)^{15} \to I^4 \to 0\ .$$

Open Question Let I be a homogeneous ideal generated by forms in degree d. Denote by $l = l(I)$, $s = n - \mathrm{depth}_{(\mathfrak{m}R)}(R)$, $c = a_*^2(R)$. By Proposition 6.3.6, there are integers $\{\alpha_{pi}\}$ and polynomials $\{Q_{\alpha_{pi}}(j)\}$ of degree $\leq l-1$ such that for $j \gg 0$ the graded minimal free resolution of I^j is

$$0 \to D_s^j \to \ldots \to D_0^j \to I^j \to 0\ ,$$

with $D_p^j = \bigoplus_i A(-\alpha_{pi} - dj)^{\beta_{pi}^j}$ and $\beta_{pi}^j = Q_{\alpha_{pi}}(j)$. In some particular cases, we have shown that this holds for $j \geq c+1$. The question is if this bound holds for any equigenerated ideal I.

6.3.2 General case

We may also study the minimal graded free resolutions of the powers of any arbitrary homogeneous ideal in the polynomial ring although in this case the asymptotic result is not so nice. Our approach will be based on a detailed study of the proof of [CHT, Theorem 3.4]. We need to introduce some notation.

Let I be a homogeneous ideal in A generated by r forms of degrees $d_1, \ldots, d_r$. Let us consider $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ the polynomial ring

with $\deg X_i = (1,0)$, $\deg Y_j = (d_j, 1)$, and let $S_2 = k[Y_1, \dots, Y_r]$. For any finitely generated bigraded module L over S_2, let us define the set

$$\delta_L(j) = \{i : L_{(i,j)} \neq 0\}.$$

Given $\underline{c} = (c_1, \dots, c_r) \in \mathbb{N}^r$, let us denote by $v(\underline{c}) = d_1 c_1 + \dots + d_r c_r$ and by $\mid \underline{c} \mid = c_1 + \dots + c_r$. Given a set $C \subset \mathbb{N}^r$, $C + \mathbb{N}^r$ denotes the set of points of $\mathbb{N}^r$ of the type $\underline{c} + \underline{c}'$ with $\underline{c} \in C$, $\underline{c}' \in \mathbb{N}^r$. Then we have:

Lemma 6.3.16 *Let L be a finitely generated bigraded S_2-module. Then there are pairs $(\alpha_i, \beta_i) \in \mathbb{Z}^2$ and finite subsets C_i of $\mathbb{N}^r$, $1 \leq i \leq m$, such that for any j*

$$\delta_L(j) = \bigcup_i \{v(\underline{c}) + \alpha_i : \underline{c} \notin C_i + \mathbb{N}^r, \mid \underline{c} \mid = j - \beta_i\}.$$

Therefore,

$$\dim_k L_{(l,j)} = \sum_{i=1}^{m} \#\{\underline{c} \in \mathbb{N}^r : \underline{c} \notin C_i + \mathbb{N}^r, \mid \underline{c} \mid = j - \beta_i, v(\underline{c}) = l - \alpha_i\}.$$

Proof. As said, the proof is based on [CHT, Theorem 3.4]. Given any finitely generated bigraded S_2-module L, there exists a sequence of bigraded submodules

$$0 = L_0 \subset L_1 \subset \dots \subset L_{m-1} \subset L_m = L$$

of L such that $M_i = L_i / L_{i-1} \cong S_2/\mathfrak{p}_i(-\alpha_i, -\beta_i)$, $1 \leq i \leq m$, with $\mathfrak{p}_i$ homogeneous prime ideals in S_2. Note that $\delta_L(j) = \bigcup_i \delta_{M_i}(j) = \bigcup_i \delta_{S_2/\mathfrak{p}_i}(j - \beta_i) + \alpha_i$, and so we can assume that L is cyclic.

Now let $L = S_2/J$, with $J \subset S_2$ a homogeneous ideal. By fixing a term order $<$ in S_2, then L has a k-basis consisting of the classes of the monomials which do not belong to the initial ideal $\mathrm{in}(J)$ of J. So we get $\delta_{S_2/J}(j) = \delta_{S_2/\mathrm{in}(J)}(j)$, and we may assume J is a monomial ideal.

Let us write $J = (Y_1^{c_{11}} \cdots Y_r^{c_{1r}}, \dots, Y_1^{c_{p1}} \cdots Y_r^{c_{pr}})$, and $\underline{c_i} = (c_{i1}, \dots, c_{ir})$ for $1 \leq i \leq p$. For any $\underline{c} \in \mathbb{N}^r$, note that $Y_1^{c_1} \cdots Y_r^{c_r} \in J$ if and only if there exists i such that $\underline{c} = \underline{c_i} + \underline{c}'$, for some $\underline{c}' \in \mathbb{N}^r$, i.e. $\underline{c} \in C + \mathbb{N}^r$, where $C = \{\underline{c}_1, \dots, \underline{c}_p\}$. Therefore

$$\delta_L(j) = \{v(\underline{c}) : \underline{c} \notin C + \mathbb{N}^r, \mid \underline{c} \mid = j\},$$

and we are done.□

Now we can show the asymptotic minimal graded free resolution of the powers of an arbitrary homogeneous ideal I.

Proposition 6.3.17 *Let I be a homogeneous ideal in the polynomial ring $A = k[X_1, \ldots, X_n]$ minimally generated by forms $f_1, \ldots, f_r$ of degrees $d_1, \ldots, d_r$. Then there are pairs $(\alpha_{pi}, \beta_{pi}) \in \mathbb{Z}^2$ and sets $C_{pi} \subset \mathbb{N}^r$, for $0 \leq p \leq s, 0 \leq i \leq k_p$, such that for j large enough the graded minimal free resolution of I^j is*

$$0 \to D_s^j \to \ldots \to D_0^j \to I^j \to 0 \ ,$$

with $D_p^j = \bigoplus_{i,\underline{c}} A(-\alpha_{pi} - v(\underline{c}))$, for α_{pi} and $\underline{c}$ such that $\underline{c} \notin C_{pi} + \mathbb{N}^r$ and $| \underline{c} |= j - \beta_{pi}$.

Proof. Let us consider the Koszul homology modules $H_p = H_p(\underline{X}, R)$ of the Rees algebra R with respect to $X_1, \ldots, X_n$. For any p, H_p is a finitely generated bigraded S_2-module with $[H_p]_{(i,j)} = \operatorname{Tor}_p^A(k, I^j)_i$. By Lemma 6.3.16, there exist $(\alpha_{pi}, \beta_{pi}) \in \mathbb{Z}^2$ and sets $C_{pi} \subset \mathbb{N}^r$ such that

$$\delta_{H_p}(j) = \bigcup_i \{v(\underline{c}) + \alpha_{pi} : \underline{c} \notin C_{pi} + \mathbb{N}^r, | \underline{c} |= j - \beta_{pi}\},$$

so we get the statement. □

Similarly to the equigenerated case, we can also prove that the rank of the modules of the graded minimal free resolution of the powers of an ideal behaves as a polynomial.

Proposition 6.3.18 *Let I be a homogeneous ideal of the polynomial ring $A = k[X_1, \ldots, X_n]$ minimally generated by forms $f_1, \ldots, f_r$ of degrees $d_1, \ldots, d_r$, and set $l = l(I)$. For any pair (α, β) and set C in the previous proposition, there exists a polynomial $Q(j)$ of degree $\leq l - 1$ such that for any $j \gg 0$*

$$Q(j) = \#\{\underline{c} : \underline{c} \notin C + \mathbb{N}^r, | \underline{c} |= j - \beta\} \ .$$

Proof. Let J be the homogeneous ideal of S_2 generated by the monomials $Y_1^{c_1} \cdots Y_r^{c_r}$ with $\underline{c} = (c_1, \ldots, c_r) \in C$. Then, denoting by $Q(j)$ the Hilbert polynomial of the graded S_2-module $L = S_2/J(-\beta)$, we have that for $j \gg 0$

$$\begin{aligned} Q(j) &= \dim_k (S_2/J)_{j-\beta} \\ &= \#\{\underline{c} : \underline{c} \notin C + \mathbb{N}^r, | \underline{c} |= j - \beta\}. \end{aligned}$$

On the other hand, since H_p is a finitely generated graded F-module of dimension $\leq l$, there exists a polynomial $P(j)$ of degree $\leq l - 1$ such that $P(j) = \dim_k [H_p]^j = \dim_k \operatorname{Tor}_p^A(k, I^j)$ for $j \gg 0$. Since $Q(j) \leq P(j)$ for $j \gg 0$, we immediately get that $\deg Q(j) \leq l - 1$. □

Remark 6.3.19 Given a homogeneous ideal I in the polynomial ring A generated by r forms of degree d, we considered its Rees algebra R with a natural bigrading. By defining $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ the polynomial ring with the bigrading $\deg X_i = (1, 0)$, $\deg Y_j = (d, 1)$, R is a bigraded finite S-module in a natural way. Now let E be any bigraded finitely generated S-module, and let consider the graded A-modules $E^j = \oplus_i E_{(i,j)}$. By taking E instead of R, we can get analogous results for the asymptotic behaviour of the A-modules E^j. In particular, by considering E to be the form ring G of I, the integral clousure of the Rees algebra $\overline{R} = \bigoplus_j \overline{I^j}$ or the symmetric algebra $\mathrm{Sym}_A(I)$ of I we have the asymptotic behaviour of I^j/I^{j+1}, $\overline{I^j}$ and $\mathrm{Sym}_j(I)$.

Bibliography

[AHT] I.M. Aberbach, C. Huneke, N.V. Trung, *Reduction numbers, Briançon–Skoda theorems and the depth of Rees rings*, Compositio Math. 97 (1995), 403–434.

[AM] R. Achilles, M. Manaresi, *Multiplicities of a bigraded ring and intersection multiplicity*, Math. Ann. 309 (1997), 573–591.

[ABW] K. Akin, D.A. Buchsbaum, J. Weyman, *Resolutions of determinantal ideals: The submaximal minors*, Adv. Math. 39 (1981), 1–30.

[ACD] A. Aramova, K. Crona, E. De Negri, *Bigeneric initial ideals, diagonal subalgebras and bigraded Hilbert functions*, preprint.

[BaSt] D. Bayer, M. Stillman, *A criterion for detecting m-regularity*, Invent. Math. 87 (1987), 1–11.

[Ber] A. Bertram, *An application of a log version of the Kodaira vanishing theorem to embedded projective varieties*, preprint.

[BEL] A. Bertram, L. Ein, R. Lazarsfeld, *Vanishing Theorems, a Theorem of Severi, and the equations defining projective varieties*, J. Amer. Math. Soc. 4 (1991), 587–602.

[BCNR] A.M. Bigatti, A. Capani, G. Niesi, L. Robbiano, *Hilbert-Poincaré Series and Elimination Problems*, preprint.

[BoSa] G. Boffi, R. Sánchez, *On the resolutions of the powers of the Pfaffian ideal*, J. Algebra 152 (1992), no. 2, 463–491.

[Bou] J.F. Boutout, *Singularités rationelles et quotients par les groupes reductifs*, Invent. Math. 88 (1987), 65–68.

[Bro] M.P. Brodmann, *The asymptotic nature of the analytic spread*, Math. Proc. Camb. Phil. Soc. 86 (1979), 35–39.

[BroSha] M.P. Brodmann, R.Y. Sharp, *Local cohomology, An algebraic introduction with geometric applications*, Cambridge studies in advanced mathematics 60, Cambridge University Press, 1998.

[Bru] W. Bruns, *Algebras Defined by Powers of Determinantal Ideals*, J. Algebra 142 (1991), 150–163.

[BH1] W. Bruns, J. Herzog, *Cohen Macaulay rings*, Cambridge studies in advanced mathematics 39, Cambridge University Press, 1993.

[BH2] W. Bruns, J. Herzog, *On computation of a-invariants*, manuscripta math. 77 (1992), 201–213.

[BV] W. Bruns, U. Vetter, *Determinantal Rings*, Lecture Notes in Mathematics, 1327, Springer–Verlag, 1988.

[Bur] L. Burch, *Codimension and analytic spread*, Proc. Camb. Phil. Soc. 72 (1972), 369–373.

[CNR] A. Capani, G. Niesi, L. Robbiano, *CoCoa, a system for doing Computations in Commutative Algebra*, 1995.

[Cha] K. Chandler, *Regularity of the powers of an ideal*, Comm. Algebra 25 (12) (1997), 3773–3776.

[Con] A. Conca, *Hilbert function and resolution of the powers of the ideal of the rational normal curve*, preprint.

[CD] A. Conca, E. De Negri, *M-sequences, graph ideals and ladder ideals of linear type*, J. Algebra 211 (1999), 599–624.

[CHTV] A. Conca, J. Herzog, N.V. Trung, G. Valla, *Diagonal subalgebras of bigraded algebras and embeddings of blow-ups of projective spaces*, Amer. J. Math. 119 (1997), 859–901.

[CV] A. Conca, G. Valla, *Hilbert function of powers of ideals of low codimension*, preprint.

[CZ] T. Cortadellas, S. Zarzuela, *On the Cohen-Macaulay property of the fiber cone of ideals with reduction number at most one*, preprint.

[CH] S.D. Cutkosky, J. Herzog, *Cohen-Macaulay coordinate rings of blowup schemes*, Comment. Math. Helv. 72 (1997), 605–617.

[CHT] S.D. Cutkosky, J. Herzog, N.V. Trung, *Asymptotic behaviour of the Castelnuovo-Mumford regularity*, Compositio Math., to appear.

[Eis] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Springer-Verlag, GTM 150, 1994.

[EG] D. Eisenbud, S. Goto, *Linear free resolutions and minimal multiplicities*, J. Algebra 88 (1984), 107–184.

[EH] D. Eisenbud, C. Huneke, *Cohen-Macaulay Rees algebras and their specializations*, J. Algebra 81 (1983), 202–224.

[ERT] D. Eisenbud, A. Reeves, B. Totaro, *Initial ideals, Veronese subrings, and rates of algebras*, Adv. Math. 109 (1994), 168–187.

[EK] S. Eliahou, M. Kervaire, *Minimal resolutions of some monomials ideals*, J. Algebra 129 (1990), 1–25.

[GG] A. Geramita, A. Gimigliano, *Generators for the defining ideal of certain rational surfaces*, Duke Math. J. 62, n§1 (1991), 61–83.

[GGH] A. Geramita, A. Gimigliano, B. Harbourne, *Projectively normal but superabundant embeddings of rational surfaces in projective spaces*, J. Algebra 169 (1994), 791–804.

[GGP] A. Geramita, A. Gimigliano, Y. Pitteloud, *Graded Betti numbers of some embedded rational n-folds*, Math. Ann. 301 (1995), 363–380.

[Gi] A. Gimigliano, *On Veronesean surfaces*, Proc. Konin. Ned. Akad. van Wetenschappen, Ser. A, 92 (1989), 71–85.

[GL] A. Gimigliano, A. Lorenzini, *On the ideal of Veronesean surfaces*, Can. J. Math. 45 (1993), 758–777.

[GH] S. Goto, S. Huckaba, *On graded rings associated to analytic deviation one ideals*, Amer. J. Math. 116 (1994), 905–919.

[GN] S. Goto, Y. Nakamura, *Cohen-Macaulay Rees algebras of ideals having analytic deviation two*, Tôhoku Math. J. 46 (1994), 573–586.

[GW1] S. Goto, K. Watanabe, *On graded rings I*, J. Math. Soc. Japan 30 (1978), 179–213.

[GW2] S. Goto, K. Watanabe, *On graded rings II*, Tokyo J. Math. 1 (1978), 237–261.

[Har] R. Hartshorne, *Algebraic Geometry*, GTM 52, Springer-Verlag, 1977.

[HHK] M. Herrmann, E. Hyry, T. Korb, *On Rees algebras with a Gorenstein Veronese subring*, J. Algebra 200 (1998), 279–311.

[HHR] M. Herrmann, E. Hyry, J. Ribbe, *On the Cohen-Macaulay and Gorenstein properties of multigraded Rees algebras*, manuscripta math. 79 (1993), 343–377.

[HHRT] M. Herrmann, E. Hyry, J. Ribbe, Z. Tang, *Reduction numbers and multiplicities of multigraded structures*, J. Algebra 197 (1997), no. 2, 311–341.

[HIO] M. Herrmann, S. Ikeda, U. Orbanz, *Equimultiplicity and blowing up*, Berlin-Heidelberg, Springer, 1988.

[HRS] M. Herrmann, J. Ribbe, P. Schenzel, *On the Gorenstein property of form rings*, Math. Z. 213 (1993), 301–309.

[HRTZ] M. Herrmann, J. Ribbe, N.V. Trung, S. Zarzuela, *Bounds for the multiplicities of almost complete intersections*, manuscripta math. 72 (1991), no. 3, 275–296.

[HRZ] M. Herrmann, J. Ribbe, S. Zarzuela, *On the Gorenstein property of Rees and form rings of powers ideals*, Trans. Amer. Math. Soc. 342 (1994), 631–643.

[HS] J. Herzog, H. Srinivasan, *Bounds for multiplicities*, Trans. Amer. Math. Soc. 350 (1998), no. 7, 2879–2902.

[HSV1] J. Herzog, A. Simis, W.V. Vasconcelos, *Approximation Complexes of Blowing-Up Rings*, J. Algebra 74 (1982), 466–493.

[HSV2] J. Herzog, A. Simis, W.V. Vasconcelos, *On the canonical module of the Rees algebra and the associated graded ring of an ideal* , J. Algebra 105 (1987), 285–302.

[HM] S. Huckaba, T. Marley, *Depth Properties of Rees Algebras and Associated Graded Rings*, J. Algebra 156 (1993), 259–271.

[Hu1] C. Huneke, *Linkage and Koszul homology of ideals*, Amer. J. Math. 104 (1982), 1043–1062.

[Hu2] C. Huneke, *The Koszul homology of an ideal*, Adv. Math. 56 (1985), 295–318.

[Hu3] C. Huneke, *The Theory of d-Sequences and Powers of Ideals*, Adv. Math. 46 (1982), 249–279.

[HuSa] C. Huneke, J. D. Sally, *Birational extensions in dimension two and integrally closed ideals*, J. Algebra 115 (1988), 481–500.

[Hy] E. Hyry, *The diagonal subring and the Cohen-Macaulay property of a multigraded ring*, Trans. Amer. Math. Soc., to appear.

[IT] S. Ikeda, N.V. Trung *When is the Rees algebra Cohen-Macaulay ?*, Comm. Algebra 17 (1989), 2893–2922.

[KMV] D. Katz, S. Mandal, J. Verma, *Hilbert functions of bigraded algebras*, in " Proceedings of the First ICTP Workshop on Commutative Algebra 1992", World Scientific, London, 1994.

[KKMS] G. Kempf, F. Knudsen, D. Mumford, B. Saint-Donat, *Toroidal embeddings*, LNM 339, Springer-Verlag, 1973.

[Ko1] V. Kodiyalam, *Homological invariants of powers of an ideal*, Proc. Amer. Math. Soc. 118 (1993), 757–764.

[Ko2] V. Kodiyalam, *Asymptotic behaviour of the Castelnuovo-Mumford regularity*, Proc. Amer. Math. Soc., to appear.

[Li] J. Lipman, *Cohen-Macaulayness in graded algebras*, Math. Res. Letters 1 (1994), 149–157.

[Mat] H. Matsumura, *Graded rings and modules*, Lecture notes in pure and applied mathematics 153, Marcel Dekker, 1993.

[RV] L. Robbiano, G. Valla, *Hilbert-Poincaré Series of Bigraded Algebras*, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) 1 (1998), no. 3, 521–540.

[Sch1] P. Schenzel, *Dualisierende Komplexe in der lokalen Algebra und Buchsbaum-Ringe*, LNM 907, Springer-Verlag, Berlin-Heidelberg-New York, 1982.

[Sch2] P. Schenzel, *Castelnuovo's index of regularity and reduction numbers*, Topics in Algebra, Banach Center Publications, Vol. 26, Part 2, PWN-Polish Scientific Publishers, Warsaw 1990.

[Sha] K. Shah, *On the Cohen-Macaulayness of the Fiber Cone of an Ideal*, J. Algebra 143 (1991), 156–172.

[STV] A. Simis, N.V. Trung, G. Valla, *The diagonal subalgebra of a blow-up algebra*, J. Pure Appl. Algebra 125 (1998), 305–328.

[Sjo] R. Sjögren, *On the regularity of graded k-algebras of Krull dimension* ≤ 1, Math. Scand. 71 (1992), 167–172.

[Swa] I. Swanson, *Powers of ideals, primary decompositions, Artin-Rees lemma and regularity*, Math. Ann. 307 (1997), 299–313.

[Tr1] N.V. Trung, *Reduction exponent and degree bound for the defining equations of graded rings*, Proc. Amer. Math. Soc. 101 (1987), 229–236.

[Tr2] N.V. Trung, *The largest non-vanishing degree of graded local cohomology modules*, preprint.

[TVZ] N.V.Trung, D.Q.Viêt, S.Zarzuela, *When is the Rees Algebra Gorenstein?* , J. Algebra 175 (1995), 137–156.

[Ul] B. Ulrich, *Lectures on Linkage and Deformation*, Workshop on Commutative Algebra, Trieste, 1992.

[Wa1] K. Watanabe, *Certain invariant subrings are Gorenstein I*, Osaka J. Math. 11 (1974), 1–8.

[Wa2] K. Watanabe, *Certain invariant subrings are Gorenstein II*, Osaka J. Math. 11 (1974), 379–388.

[Yos] Y. Yoshino, *The canonical modules of graded rings defined by generic matrices*, Nagoya Math. J. 81 (1981), 105–112.